Fixed Points of Maps on the Space of Rational Functions

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Abstract

Given integers s, t, define a function $\phi_{s,t}$ on the space of all formal series expansions by $\phi_{s,t}(\sum a_n x^n) = \sum a_{sn+t} x^n$. For each function $\phi_{s,t}$, we determine the collection of all rational functions whose Taylor expansions at zero are fixed by $\phi_{s,t}$. This collection can be described as a subspace of rational functions whose basis elements correspond to certain s-cyclotomic cosets associated with the pair (s, t).

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1 Introduction

Let $\mathfrak{R}$ denote the space of rational functions with complex coefficients. The Taylor expansion at $x = 0$ of $R \in \mathfrak{R}$ can be written as a Laurent series, i.e.,

$$R(x) \;=\; \sum_{n \gg -\infty} a_n x^n \tag{1.1}$$

where $n \gg -\infty$ denotes the fact that the coefficients vanish for large negative n. For $s, t \in \mathbb{Z}$, define the map $\phi_{s,t} : \mathfrak{R} \to \mathfrak{R}$ by

$$\phi_{s,t}(\sum a_n x^n) = \sum a_{sn+t} x^n. \tag{1.2}$$

Denote the standard s-th root of unity throughout this paper by $\omega_s = e^{2\pi i/s}$. When s is positive, consider the restriction $\phi_{s,t} : \mathfrak{R} \to \mathfrak{R}$. One can rewrite this map explicitly without the use of series expansions:

$$\phi_{s,t}(R(x)) = \left(\frac{1}{s}\right) x^{-t/s} \sum_{j=0}^{s-1} \omega_s^{-jt} R(\omega_s^j x^{1/s}). \tag{1.3}$$

Indeed, if $R(x) = \sum a_n x^n$, then $R(\omega_s^j x^{1/s}) = \sum a_n \omega_s^{jn} x^{n/s}$, and so the coefficient of $x^{(sn+t)/s}$ in the summation $\sum a_n \omega_s^{jn} x^{n/s}$ is $a_{sn+t}\omega_s^{j(sn+t)}$. Therefore, the coefficient of x^n in $\left(\frac{1}{s}\right) x^{-t/s} \sum_{j=0}^{s-1} \omega_s^{-jt} R(\omega_s^j x^{1/s})$ is $\left(\frac{1}{s}\right) \sum_{j=0}^{s-1} \omega_s^{-jt} a_{sn+t} \omega_s^{j(sn+t)} = \left(\frac{1}{s}\right) \sum_{j=0}^{s-1} a_{sn+t} = a_{sn+t}$.

The map $\phi_{2,1}$ can be used in a general procedure for the exact integration of rational functions, as described in [**?**]. Dynamical properties of $\phi_{2,1}$, including kernels of the iterates, dynamics of subclasses of rational functions, and fixed points are discussed in [?]. The purpose of this paper is to generalize one of the results in [?] by classifying, for each pair of integers s, t, the collection of all rational functions that are fixed by $\phi_{s,t}$. If s is an integer such that $s \leq 1$, then 0 is the only rational function fixed by $\phi_{s,t}$, unless, of course, $(s,t) = (1,0)$, in which case $\phi_{s,t}$ is the identity. When $s \geq 2$, however, the story is much more interesting.

2 Cyclotomic Cosets

In this section, we assume throughout that $s \geq 2$, $0 \leq t \leq s-2$, and $R \in \mathfrak{R}$ such that

$$\phi_{s,t}(R(x)) = R(x). \tag{2.1}$$

Given these restrictions on s and t, it follows that $|t/(s-1)| < 1$. Thus, if $n \leq -1$, then $n < -t/(s-1)$, and so $sn + t < n$. Assuming that $R(x)$ is fixed by $\phi_{s,t}$, we have that $a_{sn+t} = a_n$ for all n. Thus, if a_n is nonzero for any negative value of n, then there are infinitely many nonzero coefficients of negative powers of x, contradicting the assumption that $R(x)$ is of the form given in equation (??).

We write R in the form

$$R(x) = \sum_{n=0}^{\infty} f(n)x^n \tag{2.2}$$

to emphasize the fact that the coefficients can be interpreted as the images of a generating function $f : \mathbb{N} \to \mathbb{C}$. Since $R(x)$ is fixed by $\phi_{s,t}$, it follows that

$$f(n) = f(sn+t) \tag{2.3}$$

for all integers n. The following result, which was proven on page 202 of [?], elucidates the relationship between the generating function f of the coefficients of the Taylor expansion of $R(x)$ and the representation of $R(x)$ as a quotient of polynomials.

Lemma 2.1. *Let $q_1, q_2, \ldots, q_d$ be a fixed sequence of complex numbers, $d \geq 1$, and $q_d \neq 0$. The following conditions on a function $f : \mathbb{N} \to \mathbb{C}$ are equivalent:*

1. $\sum_{n\geq 0} f(n)x^n = \frac{P(x)}{Q(x)}$ *where,* $Q(x) = 1 + q_1x + q_2x^2 + q_3x^3 + \cdots + q_dx^d$.
2. *For $n \gg 0$,*

$$f(n) = \sum_{i=1}^{J} P_i(n)\lambda_i^n,$$

where $1 + q_1x + q_2x^2 + q_3x^3 + \cdots + q_dx^d = \prod_{i=1}^{J}(1-\lambda_ix)^{d_i}$, the λ_i's are distinct, and $P_i(n)$ is a polynomial in n of degree less than d_i.

In this section, we construct a collection of rational functions that are fixed by $\phi_{s,t}$, and in the next section we use the above lemma to justify that this collection spans the subspace of $\mathfrak{R}$ consisting of all rational functions that are fixed by $\phi_{s,t}$.

The description of all the fixed points of $\phi_{s,t}$ requires the notion of *cyclotomic cosets*: given n, $r \in \mathbb{N}$ with $r \geq 1$ such that r and s are relatively prime,

$$C_{s,r,n} \quad = \quad \{s^i n \bmod r : \; i \in \mathbb{Z}\} \tag{2.4}$$

is a finite set called the s-cyclotomic coset of n mod r. We will characterize the fixed points $\phi_{s,t}$ using cyclotomic cosets with a special property. To describe this property, first define

$$\beta_{s,t}(k) = t\left(\frac{s^k - 1}{s - 1}\right) \tag{2.5}$$

for which we have the following recursive formula:

$$\beta_{s,t}(j+1) = s\beta_{s,t}(j) + t. \tag{2.6}$$

Definition 2.2. A positive integer r is called *distinguished* with respect to the pair (s, t) if r and s are relatively prime and

$$r \mid \beta_{s,t}(\mathrm{Ord}(s; r)), \tag{2.7}$$

where $\mathrm{Ord}(s; r)$ represents the smallest positive integer i such that $s^i \equiv 1 \mod r$. We say $r = 0$ is distinguished with respect to (s, t) if and only if $t = 0$. We denote the set of integers distinguished with respect to (s, t) by $\Omega(s, t)$.

Proposition 2.3. *For each pair (s, t), the set $\Omega(s, t)$ is infinite.*

Proof. Since $\Omega(s, t) \subset \Omega(s, 1)$, we need only show that $\Omega(s, 1)$ is infinite. Let r be a positive integer such that $\gcd(r, s(s - 1)) = 1$. If $\alpha = \mathrm{Ord}(s; r)$, then $s^\alpha \equiv 1 \mod r$; that is, $r \mid s^{\mathrm{Ord}(s;r)} - 1$. Since $s^{\mathrm{Ord}(s;r)} - 1$ is a multiple of $(s - 1)$, and r is relatively prime to $(s - 1)$, it follows that $r(s - 1) \mid s^{\mathrm{Ord}(s;r)} - 1$. Thus, $r \mid \frac{s^{\mathrm{Ord}(s;r)}-1}{s-1} = \beta_{(s,1)}(\mathrm{Ord}(s; r))$, and so r is distinguished with respect to $(s, 1)$. $\square$

For example, consider

$$\Omega(3, 1) = \{1, 4, 5, 7, 10, 11, 13, 14, 17, 19, 20, 23, 25, 28, 29, 31, 34, 35, 37, 38, \ldots\}.$$

From Proposition **??**, we see that $\Omega(3, 1)$ contains the arithmetic sequences $\{6n + 1\}$ and $\{6n + 5\}$. With a little more effort, one can show that $\Omega(3, 1)$ also contains the arithmetic sequences $\{24n+4\}$, $\{24n+10\}$, $\{24n+14\}$, and $\{24n+20\}$. The smallest integer in $\Omega(3, 1)$ not contained in any of these sequences is 40. Moreover, a calculation shows that $96n + 40$, for $0 \le n \le 5$ is in $\Omega(3, 1)$, but $616 = 96 \cdot 6 + 40$ is not in $\Omega(3, 1)$. An interesting question of further study is whether the sets $\Omega(s, t)$ have a nice characterization. For example, we might ask whether they can be written as a (possibly infinite) union of arithmetic sequences, as is the case for $\Omega(2, 1)$, which consists precisely of all odd natural numbers. However, the example $\Omega(3, 1)$ suggests that this may not be the case in general.

A generating set for the collection of fixed points of $\phi_{s,t}$ will be indexed by s-cyclotomic cosets $C_{s,r,n}$ where r is distinguished with respect to (s, t). Note that by computing

$$\phi_{s,t}\left(\frac{1}{1 - \lambda x}\right) = \frac{\lambda^t}{1 - \lambda^s x} \tag{2.8}$$

we acquire the following formula for the iterates of $\phi_{s,t}$:

$$\phi^{(k)}_{s,t}\left(\frac{1}{1 - \lambda x}\right) = \frac{\lambda^{\beta_{s,t}(k)}}{1 - \lambda^{s^k} x}. \tag{2.9}$$

For $r \geq 1$ and $n \in \mathbb{N}$, define

$$\psi_{s,t,r,n}(x) = \sum_{j=1}^{\operatorname{Ord}(s;r)} \frac{\omega_r^{n\beta_{s,t}(j)}}{1 - \omega_r^{ns^j} x} = \sum_{j=1}^{\operatorname{Ord}(s;r)} \phi_{s,t}^{(j)} \left(\frac{1}{1 - \omega_r^n x} \right). \tag{2.10}$$

Note that if $n = 0$, then $\psi_{s,t,r,0} = 1/(1-x)$. If $t = 0$, then $r = 0$ is distinguished with respect to (s,t), and we define

$$\psi_{s,0,0,n}(x) = 1. \tag{2.11}$$

Proposition 2.4. *If r is distinguished with respect to (s,t), then $\psi_{s,t,r,n}(x)$ is fixed by $\phi_{s,t}$.*

Proof. If $r > 1$ is distinguished with respect to (s,t), then

$$\phi_{s,t}^{(\operatorname{Ord}(s;r)+1)} \left(\frac{1}{1 - \omega_r^n x} \right) = \phi_{s,t} \left(\phi_{s,t}^{(\operatorname{Ord}(s;r))} \left(\frac{1}{1 - \omega_r^n x} \right) \right) = \phi_{s,t} \left(\frac{1}{1 - \omega_r^n x} \right),$$

and so

$$\phi_{s,t}(\psi_{s,t,r,n}(x)) = \phi_{s,t} \left(\sum_{j=1}^{\operatorname{Ord}(s;r)} \phi_{s,t}^{(j)} \left(\frac{1}{1 - \omega_r^n x} \right) \right) = \left(\sum_{j=1}^{\operatorname{Ord}(s;r)} \phi_{s,t}^{(j+1)} \left(\frac{1}{1 - \omega_r^n x} \right) \right) = \psi_{s,t,r,n}(x).$$

Thus $\psi_{s,t,r,n}(x)$ is fixed by $\phi_{s,t}$. Since constants are fixed by $\phi_{s,0}$, it follows that $\psi_{s,0,0,n}$ is fixed by $\phi_{s,0}$. Since $r = 0$ is distinguished only with respect to $t = 0$, we have shown the result holds in all possible cases. □

3 The Space of Fixed Points of $\phi_{s,t}$

We now classify all the fixed points of $\phi_{s,t}$ for all integers s, t. To do so, we first demonstrate a bijective correspondence between fixed points of $\phi_{s,t}$ and $\phi_{s,t+u(s-1)}$ where u is an arbitrary integer.

Lemma 3.1. *For all integers s, t, u, the rational function $R(x)$ is a fixed point of $\phi_{s,t}$ iff $x^{-u}R(x)$ is a fixed point of $\phi_{s,t+u(s-1)}$.*

Proof. Using equation (**??**), one can show directly that for any integers s, t, u,

$$\phi_{s,t}(R(x)) = x^u \phi_{s,t+(s-1)u}(x^{-u}R(x)),$$

and so

$$\begin{aligned} \phi_{s,t}(R(x)) = R(x) \quad &\Leftrightarrow \quad x^u \phi_{s,t+(s-1)u}(x^{-u}R(x)) = R(x) \\ &\Leftrightarrow \quad \phi_{s,t+(s-1)u}(x^{-u}R(x)) = x^{-u}R(x). \end{aligned}$$

□

Given this correspondence, we only have to compute the fixed points of $\phi_{s,t}$ in case $0 \leq t \leq s - 2$. Once this is accomplished, to compute the fixed points of $\phi_{s,t}$ for arbitrary t, we only need to find t', u such that $0 \leq t' \leq s - 2$ and $t = t' + u(s-1)$, and then use the correspondence. The following result provides the missing component of this scheme, thus allowing us to compute the fixed points $\phi_{s,t}$ for any integers s and t.

Proposition 3.2. *Suppose $s \geq 2$ and $0 \leq t \leq s-2$. A rational function is fixed by $\phi_{s,t}$ if and only if it is a linear combination of the functions $\psi_{s,t,r,n}(x)$ where r is distinguished with respect to (s,t) and n is relatively prime to r.*

Proof. We showed in Proposition **??** that if r is distinguished with respect to (s,t), then $\psi_{s,t,r,n}(x)$ is fixed by $\phi_{s,t}$, and so every linear combination of such functions must be fixed by $\phi_{s,t}$.

To prove the converse, we consider a rational function $R(x)$ fixed by $\phi_{s,t}$, and express it as

$$R(x) = C(x) + \frac{P(x)}{Q(x)} \tag{3.1}$$

where $C(x), P(x), Q(x)$ are polynomials such that $P(x)$ and $Q(x)$ are relatively prime with $\deg P(x) < \deg Q(x)$. Our first goal is to show that the poles of $R(x)$ must be simple. We write

$$\frac{P(x)}{Q(x)} = \sum_{n=0}^{\infty} f(n)x^n \tag{3.2}$$

where $f(n)$ is the generating function for $P(x)/Q(x)$. Since $f(n) = f(sn+t)$, we have by Lemma **??**, $f(sn+t) = \sum P_i(sn+t)\lambda_i^t(\lambda_i^s)^n$ and

$$Q(x) = \prod_{i=1}^{J}(1-\lambda_i x)^{d_i} = \prod_{i=1}^{J}(1-\lambda_i^s x)^{e_i}, \tag{3.3}$$

and so

$$\{\lambda_1, \ldots, \lambda_J\} = \{\lambda_1^s, \ldots, \lambda_J^s\}. \tag{3.4}$$

Thus the set $\{\lambda_1, \cdots, \lambda_J\}$ is permuted by the map $z \mapsto z^s$, and so each λ_j is a primitive r_j-th root of unity where r_j is a positive integer. Moreover, since $\{\lambda_1, \ldots, \lambda_J\}$ is permuted by the map $z \mapsto z^s$, it follows that for each $1 \leq j \leq J$, there exists a positive integer ℓ such that $\lambda_j^{s\ell} = \lambda_j$ (after applying the map $z \mapsto z^s$ multiple times). Therefore, $\lambda_j^{s\ell-1} = 1$, and so $r_j \mid s\ell - 1$. Thus r_j and s are relatively prime.

Let $M = \operatorname{lcm}(r_1, \ldots, r_j)$ and for $a \in \mathbb{N}$, define

$$R_a = \{m \in \mathbb{N} : m \equiv a \mod M\}.$$

Let $f_a = f\Big|_{R_a}$ be the restriction of the function $f : \mathbb{N} \to \mathbb{C}$ to the set R_a. Then

$$f_a(a+jM) = \sum_{i=1}^{J} P_i(a+jM)\lambda_i^{a+jM} = \sum_{i=1}^{J} P_i(a+jM)\lambda_i^a,$$

and so each f_a has a representation as a polynomial in the variable j since λ_i^a is constant on the set R_a. We denote the natural extension of this map to an element of the polynomial ring $\mathbb{C}[j]$ by F_a. Note that the restriction of F_a to $\mathbb{N}$ need not be f in general. Our goal is to prove that each F_a is a constant function, with corresponding constant denoted by c_a. Once this is shown, we have

$$\frac{P(x)}{Q(x)} = \sum_{n=0}^{\infty} f(n)x^n = \sum_{a=0}^{M-1} c_a \sum_{j=0}^{\infty} x^{a+jM} = \sum_{a=0}^{M-1} \frac{c_a x^a}{1-x^M}, \tag{3.5}$$

and so $P(x)/Q(x)$ is a rational function with only simple poles, as desired.

It remains to show that each polynomial map $F_a : \mathbb{C} \to \mathbb{C}$ is a constant function. For each positive integer n, define

$$S_n = \{\beta^{(j)}_{s,t}(n) : j \in \mathbb{N}\}. \tag{3.6}$$

We say that a has an *infinite cross-section* if $R_a \cap S_n$ is an infinite set for some $n \in \mathbb{N}$. We proceed by considering two cases, depending on whether a has an infinite cross-section or not

Case 1: Suppose a has an infinite cross-section, i.e., $R_a \cap S_n$ is an infinite set. Since $f(j) = f(sj + t)$ for all $j \in \mathbb{N}$, F_a is constant on $R_a \cap S_n$. Since $R_a \cap S_n$ is an infinite set, F_a is a constant polynomial.

Case 2: Suppose a does not have an infinite cross-section, i.e., $R_a \cap S_n$ is finite for all positive integers n. Then $R_a \cap S_n$ must be nonempty for infinitely many values of n. Since there are only finitely many distinct sets of the form R_b, it follows that for each S_n, there exists $b \in \mathbb{N}$ such that $R_b \cap S_n$ is infinite. Moreover, since there are only finitely many choices for R_b, there is at least one $b \in \mathbb{N}$ such that there exist infinitely many values of n where $R_a \cap S_n$ is nonempty and $R_b \cap S_n$ is infinite. Since b has an infinite cross-section, an application of Case 1 demonstrates that the restriction of f to R_b is the constant function c_b. Since f is constant on each S_n, the restriction of f to S_n is the constant c_b. Thus F_a achieves the value c_b infinitely many times, and so F_a must be a constant polynomial.

Thus in either case, we have that F_a is a constant polynomial, and so the poles of R must be simple. Using this fact, we can decompose $R(x)$ using partial fractions:

$$R(x) = C(x) + \sum_{j=1}^{J} \frac{\alpha_j}{1 - \lambda_j x}. \tag{3.7}$$

Via (**??**), an application of $\phi_{s,t}$ yields

$$R(x) = \phi_{s,t}(R(x)) = \phi_{s,t}(C(x)) + \sum_{j=1}^{J} \frac{\alpha_j \lambda_j^t}{1 - \lambda_j^s x}. \tag{3.8}$$

Each rational function has a unique decomposition, and since $\phi_{s,t}$ maps polynomials to polynomials,

$$C(x) = \phi_{s,t}(C(x)) \tag{3.9}$$

and

$$\sum_{j=1}^{J} \frac{\alpha_j}{1 - \lambda_j x} = \sum_{j=1}^{J} \frac{\alpha_j \lambda_j^t}{1 - \lambda_j^s x} = \phi_{s,t}\left(\sum_{j=1}^{J} \frac{\alpha_j}{1 - \lambda_j x}\right). \tag{3.10}$$

If $t > 0$, it is easy to see that no nonzero polynomial is fixed by $\phi_{s,t}$, in which case $C(x) = 0$. If $t = 0$, then the only polynomials fixed by $\phi_{s,t}$ are constant, and so $C(x)$ is a constant multiple of $\psi_{s,0,0,n} = 1$.

Now we only have left to show that the second summand in (**??**) is a linear combination of functions of the form $\psi_{s,t,r,n}$. To do this, we begin by showing that each r_k is distinguished with respect to (s,t). We have already shown that r_k and s are relatively prime for each k. Using (**??**), multiple iterations of $\phi_{s,t}$ to (**??**) yield

$$\sum_{j=1}^{J} \frac{\alpha_j}{1-\lambda_j x} = \phi_{s,t}^{(\mathrm{Ord}(s;r_k))}\left(\sum_{j=1}^{J} \frac{\alpha_j}{1-\lambda_j x}\right) = \sum_{j=1}^{J} \frac{\alpha_j \lambda_j^{\beta_{s,t}(\mathrm{Ord}(s;r_k))}}{1-\lambda_j^{s^{\mathrm{Ord}(s;r_k)}} x}. \tag{3.11}$$

The term corresponding to $j=k$ in the first of these three expressions is

$$\frac{\alpha_k}{1-\lambda_k x}, \tag{3.12}$$

and the corresponding term in the last of these three expressions is

$$\frac{\alpha_k \lambda_k^{\beta_{s,t}(\mathrm{Ord}(s;r_k))}}{1-\lambda_k^{s^{\mathrm{Ord}(s;r_k)}} x} = \frac{\alpha_k \lambda_k^{\beta_{s,t}(\mathrm{Ord}(s;r_k))}}{1-\lambda_k x}. \tag{3.13}$$

Thus

$$\lambda_k^{\beta_{s,t}(\mathrm{Ord}(s;r_k))} = 1. \tag{3.14}$$

Therefore, $r_k \mid \beta_{s,t}(\mathrm{Ord}(s;r_k))$, and so r_k is distinguished with respect to (s,t).

Now that we've shown that each r_k is distinguished with respect to (s,t), group terms in the sum

$$\sum_{j=1}^{J} \frac{\alpha_j}{1-\lambda_j x} \tag{3.15}$$

according to the orbits of the map $z \mapsto z^s$ on the set $\{\lambda_1, \ldots, \lambda_J\}$. Since $r_k \mid \beta_{s,t}(\mathrm{Ord}(s;r_k))$ for each k, we know that the sum of terms in (**??**) corresponding to a single orbit must be of the form

$$\mathcal{O}(k) = \sum_{i=1}^{m} \phi_{s,t}^{(i)}\left(\frac{\alpha_k}{1-\lambda_k x}\right), \tag{3.16}$$

where m is the length of the orbit of λ_k under the map $z \mapsto z^s$. That is, m is the smallest positive integer such that $\lambda_k^{s^m} = 1$, and so $m = \mathrm{Ord}(s;r_k)$. Moreover, λ_k is a primitive r_k-th root of unity, and so it must be of the form $\lambda_k = (\omega_{r_k})^n$ for some $n \in \mathbb{N}$ such that r_k and n are relatively prime. Thus

$$\mathcal{O}(k) = \alpha_k \left(\sum_{i=1}^{\mathrm{Ord}(s;r_k)} \phi_{s,t}^{(i)}\left(\frac{1}{1-\omega_{r_k}^n x}\right)\right) = \alpha_k \psi_{s,t,r_k,n}(x), \tag{3.17}$$

and so (**??**), and hence (**??**), is a linear combination of rational functions of the form $\psi_{s,t,r,n}$. $\square$

It turns out that the collection of rational functions of the form $\psi_{s,t,r,n}$ does not form a basis of fixed points. The lemma below shows that there is redundancy in the collection. Since cyclotomic cosets have many different representations, we must compare the ways in which $\psi_{s,t,r,n}$ and $\psi_{s,t,r,n'}$ are defined for two distinct representations $C_{s,r,n}$ and $C_{s,r,n'}$ of the same coset. Although we have not defined $\psi_{s,t,r,n}$ to be invariant with respect to different representations, they will be the same up a constant multiple.

Lemma 3.3. *If $C_{s,r,n} = C_{s,r,n'}$, then $\psi_{s,t,r,n}$ and $\psi_{s,t,r,n'}$ are scalar multiples of one another.*

Proof. With the aid of (**??**), we compute

$$\begin{aligned}
\psi_{s,t,r,ns}(x) &= \sum_{j=1}^{\mathrm{Ord}(s;r)} \frac{\omega_r^{ns\beta_{s,t}(j)}}{1-\omega_r^{snsj}x} \\
&= \sum_{j=1}^{\mathrm{Ord}(s;r)} \frac{\omega_r^{n(\beta_{s,t}(j+1)-t)}}{1-\omega_r^{ns^{j+1}}x} \\
&= \omega_r^{-nt} \sum_{j=1}^{\mathrm{Ord}(s;r)} \frac{\omega_r^{n\beta_{s,t}(j+1)}}{1-\omega_r^{ns^{j+1}}x} \\
&= \omega_r^{-nt} \sum_{j=1}^{\mathrm{Ord}(s;r)} \phi_{s,t}^{(j+1)}\left(\frac{1}{1-\omega_r^{n}x}\right) \\
&= \omega_r^{-nt}\phi_{s,t}(\psi_{s,t,r,n}(x)) \\
&= \omega_r^{-nt}\psi_{s,t,r,n}(x).
\end{aligned}$$

Thus ψ_{s,t,r,s^in} and $\psi_{s,t,r,n}$ are scalar multiples of one another for all $i \in \mathbb{N}$. If $C_{s,r,n} = C_{s,r,n'}$, then for some $i \in \mathbb{N}$, we have $n' \equiv s^i n \mod r$. By this equivalence, $\psi_{s,t,r,n'} = \psi_{s,t,r,s^in}$, and so the result follows. □

Using Lemma **??**, we can show that if two of functions of the form $\psi_{s,t,r,n}$ have a pole in common, then they are actually the same up to a scalar multiple. The following lemma leads us this result.

Lemma 3.4. *Suppose r_i is a positive integer that is distinguished with respect to (s,t), and n_i is a positive integer relatively prime to r_i for $i = 1,2$. If ψ_{s,t,r_1,n_1} and ψ_{s,t,r_2,n_2} have a pole in common, then $r_1 = r_2$ and $C_{s,r_1,n_1} = C_{s,r_2,n_2}$.*

Proof. Note that $\psi_{s,t,r,n}$ has poles at $\omega_r^{-ns^j}$ for $0 \le j \le \mathrm{Ord}(s;r)$; that is, $\psi_{s,t,r,n}$ has poles at ω_r^{-c} where $c \in C_{s,r,n}$. Suppose ψ_{s,t,r_1,n_1} and ψ_{s,t,r_2,n_2} have a pole in common; that is, $e^{-2\pi i c_1/r_1} = e^{-2\pi i c_2/r_2}$, where $c_i \in C_{s,r_i,n_i}$. Thus, $c_1/r_1 - c_2/r_2 \in \mathbb{Z}$. Without loss of generality, we can choose $1 \le c_i < r_i$, in which case $0 < c_1/r_1 < 1$, and so $c_1/r_1 = c_2/r_2$. Since $\gcd(r_i, n_i) = 1$ and $c_i = s^{j_i}n_i \mod r_i$ for some $j_i \in \mathbb{N}$, it follows that c_i and r_i are relatively prime, and so $c_1 = c_2$ and $r_1 = r_2$. Therefore, $s^{j_1}n_1 = s^{j_2}n_2 \mod r$ (where $r = r_1 = r_2$), and so $C_{s,r,n_1} = C_{s,r,n_2}$. □

We now precisely describe the redundancy in the collection $\{\psi_{s,t,r,n}\}$ for fixed s and t. We begin by defining an equivalence relation $\sim_{s,r}$ on $(C_{s,r,n} - \{0\})$ by $n_1 \sim_{s,r} n_2$ if $C_{s,r,n_1} = C_{s,r,n_2}$. Let $\Lambda_{s,r}$ be a collection of coset representatives (all chosen to be less than r) of $(C_{s,r,n} - \{0\})/\sim_{s,r}$. That is, $\Lambda_{s,r}$ is maximal set consisting of positive integers such that no two are in the same cyclotomic coset.

Theorem 3.5. *Suppose $s \ge 2$ and $0 \le t \le s-2$. The function $1/(1-x)$ together with the collection of all $\psi_{s,t,r,n}$ where r is distinguished with respect to (s,t) and $n \in \Lambda_{s,r}$ form a basis for the set of all rational functions that are fixed points of $\phi_{s,t}$.*

Proof. The case $n = 0$ corresponds to the function $1/(1-x)$. We now consider the case $n > 0$. Given an integer r that is distinguished with respect to (s, t), and an integer n that is relatively prime to r, there exists $n' \in \Lambda_{s,r}$ such that $C_{s,r,n} = C_{s,r,n'}$, in which case by Lemma **??**, $\psi_{s,t,r,n}$ and $\psi_{s,t,r,n'}$ are scalar multiples of one another. Thus, by Proposition **??**, this collection spans the space of rational functions fixed by $\phi_{s,t}$.

Suppose ψ_{s,t,r_1,n_1} and ψ_{s,t,r_2,n_2} have a pole in common where $n_i \in \Lambda_{s,r_i}$. Then by Lemma **??**, $r_1 = r_2$ and $C_{s,r_1,n_1} = C_{s,r_2,n_2}$. Thus by the definition of $\Lambda_{s,r_1} = \Lambda_{s,r_2}$, $n_1 = n_2$. Therefore, none of the elements of the collection have a pole in common, and so no nontrivial linear combination of elements of this collection can be zero. $\square$

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