COUNTING STAIRCASES IN INTEGER COMPOSITIONS

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ABSTRACT. The main theorem establishes the generating function F which counts the number of times the staircase $1^+2^+3^+\cdots m^+$ fits inside an integer composition of n.

$$F = \frac{k_m - \frac{qx^my}{1-x}k_{m-1}}{(1-q)x^{\binom{m+1}{2}}\left(\frac{y}{1-x}\right)^m + \frac{1-x-xy}{1-x}\left(k_m - \frac{qx^my}{1-x}k_{m-1}\right)}.$$

where

$$k_m = \sum_{\text{æ}=0}^{m-1} x^{mj-\binom{j}{2}}\left(\frac{y}{1-x}\right)^j.$$

Here x and y respectively track the composition size and number of parts, whilst q tracks the number of such staircases contained.

1. INTRODUCTION

In several recent papers the notion of integer compositions of n (represented as the associated bargraph) have been used to model certain problems in physics. See for example [2, 7–9] where bargraphs are a representation of a polymer at an adsorbing wall subject to several forces.

In a paper by a current author et al (see [1]), the x-ray process was modelled using permutation matrices as a two dimensional analogue of the object being x-rayed, where the examining rays are modelled by diagonal lines with equation $x + y = n$ for positive integers n. The current paper is based instead on integer compositions as the object analogue and where the examining rays are represented by equation $x - y = n$ for non negative integers n. Since this model is essentially parameterized by the degree to which the x-rays are contained inside an arbitrary composition, it translates naturally to obtaining a generating function which tracks the number of "staircases" which are contained inside particular integer compositions of n. More precisely, we will obtain a generating function which counts (with the exponent s of q as tracker) the number of times the staircase $1^+2^+3^+\cdots m^+$ (m fixed) fits inside particular compositions. So the term of our generating function $n(a,b,s)x^ay^bq^s$ indicates that there are in total $n(a,b,s)$ compositions of a with b parts in which the staircases $1^+2^+3^+\cdots m^+$ occurs exactly s times.

Date: October 27, 2016.

1991 *Mathematics Subject Classification.* 05A18, 05A15, 15A06, 15A09.

Key words and phrases. composition, generating function.

1.1. **Definitions.** A *composition of a positive integer n* is a sequence of k positive integers $a_1, a_2, \cdots a_k$, each called a part such that $n = \sum_{ß=1}^{k} a_i$; A *staircase* $1^+2^+3^+ \cdots m^+$ is a word with m sequential parts from left to right where for $1 \leq i \leq m$ the ith part $\geq i$.

See for example the staircase in Figure 1 below.

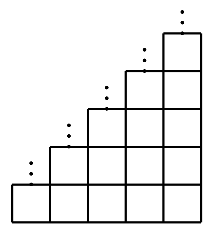

FIGURE 1. The staircase $1^+2^+3^+4^+5^+$

Much recent work has been done on various statistics relating to compositions. See, for example, [3,5,6] and [4] and references therein.

A particular composition may be represented as a bargraph (see [4] and [2]). For example the composition $4+3+1+2+3$ of 13 represented in Figure 2 as a bargraph, contains exactly one $1^+2^+3^+$ staircase, three 1^+2^+ staircases and five 1^+ staircases. It contains no others.

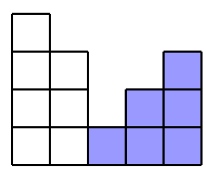

FIGURE 2. The composition $4+3+1+2+3$ containing one staircase $1^+2^+3^+$ (coloured) and three 1^+2^+ staircases

In this paper, compositions (ie their associated bargraphs) are the analogue for a (2-dimensional) object to be x-rayed (as explained above). Across all possible compositions, the shapes are parameterized in a generating function by a marker variable q which tracks the number of $1^+2^+3^+ \cdots m^+$ staircases (again with m fixed) that fit inside a composition. The generating function in question is defined as

$$F = \sum_{a\geq 1; b\geq 1; s\geq 0} n(a,b,s)x^a y^b q^s, \tag{1}$$

where $n(a,b,s)$ is the number of compositions of a with b parts that contain s staircases $1^+2^+3^+ \cdots m^+$.

The main theorem arrived at by the end of the paper consists in establishing a formula for the generating function F defined in equation (1). We state it here for completeness:

$$F = \frac{k_m - \frac{qx^m y}{1-x} k_{m-1}}{(1-q)x^{\binom{m+1}{2}} \left(\frac{y}{1-x}\right)^m + \frac{1-x-xy}{1-x}\left(k_m - \frac{qx^m y}{1-x} k_{m-1}\right)},$$

where $k_m = \sum_{\text{æ}=0}^{m-1} x^{mj-\binom{j}{2}} \left(\frac{y}{1-x}\right)^j$. Prior to this main theorem, several lemmas present a set of recursions which are used in proving this result.

2. Proofs

2.1. **Warmup: compositions containing words of the form 1^+2^+ or $1^+2^+3^+$.** Consider words which are of the form 1^+2^+; i.e., words of two parts adjacent to each other from left to right with the first being a letter > 0 and the second being a letter > 1.

We let F be the generating function for all words; F_a be the generating function for all words starting with the letter a and in general $F_{a_1a_2\cdots a_n}$ be the gf (generating function) for words starting with the letters $a_1a_2\cdots a_n$. So by definition

$$F = 1 + \sum_{a\geq 1} F_a. \tag{2}$$

And we have the following recurrence:

$$F_a = x^a y + F_{a1} + F_{a2} + F_{a3} + \cdots \tag{3}$$

Now $F_{a1} = x^a y F_1$ and $F_{ab} = qx^a y F_b$ for $b > 1$. So $F_a = x^a y(1 + F_1 + qF_2 + qF_3 + \cdots)$. Thus for all $a \geq 1$, we have $F_a = x^a y(1-q)(1+F_1) + qx^a yF$. As the second part of our warmup, we now examine the pattern $1^+2^+3^+$, i.e., we focus on compositions which contain this word sequence.

Extracting part of the first letter, we have

$$F_a = x^{a-1} F_1. \tag{4}$$

From equation (2),

$$F = 1 + \sum_{a\geq 1} F_a = 1 + \frac{1}{1-x} F_1. \tag{5}$$

Also

$$\begin{aligned} F_1 &= xy + (F_{11} + F_{12} + F_{13} + \cdots) \\ &= xy + xy(F_1 + F_{12} + xF_{12} + x^2 F_{12} + \cdots) \\ &= xy + xyF_1 + \frac{1}{1-x} F_{12}, \end{aligned} \tag{6}$$

where

$$\begin{aligned} F_{12} &= x^3y^2 + F_{121} + F_{122} + (F_{123} + \cdots) \\ &= x^3y^2 + x^3y^2F_1 + x^2yF_{12} + (qx^3yF_{12} + qx^4yF_{12} + \cdots) \\ &= x^3y^2 + x^3y^2F_1 + x^2yF_{12} + \frac{qx^3y}{1-x}F_{12}. \end{aligned} \tag{7}$$

The last three equations have three unknowns F, F_1, and F_{12} which we can solve for F using Cramer's rule. However, instead, we try the general pattern.

2.2. **The general pattern** $1^+2^+3^+\cdots m^+$**.** As before, $F_a = x^{a-1}F_1$ and

$$F = 1 + \sum_{a\geq 1} F_a = 1 + \frac{1}{1-x}F_1. \tag{8}$$

Now

$$\begin{aligned} F_1 &= xy + (F_{11} + F_{12} + F_{13} + \cdots) \\ &= xy + xy(F_1 + F_{12} + xF_{12} + x^2F_{12} + \cdots) \\ &= xy + xyF_1 + \frac{1}{1-x}F_{12} \end{aligned} \tag{9}$$

and

$$\begin{aligned} F_{12} &= x^3y^2 + F_{121} + F_{122} + (F_{123} + \cdots) \\ &= x^3y^2 + x^3y^2F_1 + x^2yF_{12} + (F_{123} + xF_{123} + x^2F_{123} + \cdots) \\ &= x^3y^2 + x^3y^2F_1 + x^2yF_{12} + \frac{1}{1-x}F_{123}. \end{aligned} \tag{10}$$

Next, by a similar process

$$F_{123} = x^6y^3 + x^6y^3F_1 + x^5y^2F_{12} + x^3yF_{123} + \frac{1}{1-x}F_{1234}. \tag{11}$$

Proceeding in this way, we obtain in general for all $j \leq m-1$

$$\begin{aligned} F_{12\cdots j} &= x^{\binom{j+1}{2}}y^j + x^{\binom{j+1}{2}-\binom{1}{2}}y^jF_1 + x^{\binom{j+1}{2}-\binom{2}{2}}y^{j-1}F_{12} \\ &\quad + x^{\binom{j+1}{2}-\binom{3}{2}}y^{j-2}F_{123} + \cdots + x^{\binom{j+1}{2}-\binom{j}{2}}yF_{12\cdots j} + \frac{1}{1-x}F_{12\cdots j+1}. \end{aligned} \tag{12}$$

with

$$F_{12\cdots m} = qx^myF_{12\cdots m-1}. \tag{13}$$

To simplify the presentation we put $z = \frac{-1}{1-x}$. Now, we rewrite equations (7)-(13) in matrix form. So we first define the matrix $\mathbf{A}$ as

$$\begin{pmatrix}
1 & z & 0 & 0 & \cdots & \cdots & \cdots & 0\\
0 & 1-x^{\binom{2}{2}-\binom{1}{2}}y & z & 0 & \cdots & \cdots & \cdots & 0\\
0 & -x^{\binom{3}{2}-\binom{1}{2}}y^2 & 1-x^{\binom{3}{2}-\binom{2}{2}}y & z & \cdots & & & 0\\
\vdots & & & & & & & \vdots\\
0 & -x^{\binom{m-1}{2}-\binom{1}{2}}y^{m-2} & -x^{\binom{m-1}{2}-\binom{2}{2}}y^{m-3} & -x^{\binom{m}{2}-\binom{3}{2}}y^{m-4} & \cdots & -x^{\binom{m-1}{2}-\binom{m-2}{2}}y & z & 0\\
0 & -x^{\binom{m}{2}-\binom{1}{2}}y^{m-1} & -x^{\binom{m}{2}-\binom{2}{2}}y^{m-2} & -x^{\binom{m}{2}-\binom{3}{2}}y^{m-3} & \cdots & -x^{\binom{m}{2}-\binom{m-2}{2}}y^2 & 1-x^{\binom{m}{2}-\binom{m-1}{2}}y & z\\
0 & 0 & 0 & 0 & \cdots & 0 & -qx^my & 1
\end{pmatrix}$$

and $\mathbf{C}$ to be the vector $\left(x^{\binom{1}{2}}, x^{\binom{2}{2}}y, x^{\binom{3}{2}}y^2, \cdots, x^{\binom{m-1}{2}}y^{m-2}, x^{\binom{m}{2}}y^{m-1}, 0\right)^T$. Then the matrix form of our equations is $\mathbf{AX} = \mathbf{C}$ where it is the first entry of matrix $\mathbf{X}$ (the matrix of variables from equations (7)-(13)) that is our required generating function F. So defining $\mathbf{B}$ as the matrix obtained from the above matrix $\mathbf{A}$ by replacing its first column with the entries from $\mathbf{C}$; i.e.

$$\begin{pmatrix}
x^{\binom{1}{2}} & z & 0 & \cdots & \cdots & \cdots & 0\\
x^{\binom{2}{2}}y & 1-x^{\binom{2}{2}-\binom{1}{2}}y & z & \cdots & \cdots & \cdots & 0\\
x^{\binom{3}{2}}y^2 & -x^{\binom{3}{2}-\binom{1}{2}}y^2 & 1-x^{\binom{3}{2}-\binom{2}{2}}y & \cdots & & & 0\\
\vdots & & & & & & \vdots\\
x^{\binom{m-1}{2}}y^{m-2} & -x^{\binom{m-1}{2}-\binom{1}{2}}y^{m-2} & -x^{\binom{m-1}{2}-\binom{2}{2}}y^{m-3} & \cdots & -x^{\binom{m-1}{2}-\binom{m-2}{2}}y & z & 0\\
x^{\binom{m}{2}}y^{m-1} & -x^{\binom{m}{2}-\binom{1}{2}}y^{m-1} & -x^{\binom{m}{2}-\binom{2}{2}}y^{m-2} & \cdots & -x^{\binom{m}{2}-\binom{m-2}{2}}y^2 & 1-x^{\binom{m}{2}-\binom{m-1}{2}}y & z\\
0 & 0 & 0 & \cdots & 0 & -qx^my & 1
\end{pmatrix}.$$

By Cramer's rule, we obtain

$$F = \frac{\det \mathbf{B}}{\det \mathbf{A}}. \tag{14}$$

2.3. **Equations for** $\det \mathbf{A}$ **and** $\det \mathbf{B}$ **in a form that can be solved recursively.** Define the mxm matrix $\mathbf{N_m}$, to be the first m rows and columns of the $(m+1)$x$(m+1)$ matrix $\mathbf{A}$, but where the first column of $\mathbf{A}$ has initially been replaced by the first m entries of $\mathbf{C}$. To simplify the notation further, we let $w_{ij} = x^{\binom{i}{2}-\binom{j}{2}}y^{i-j}$ and so explicitly written out,

$$\mathbf{N_m} := \begin{pmatrix}
x^{\binom{1}{2}}y^0 & z & 0 & 0 & \cdots & 0\\
x^{\binom{2}{2}}y & 1-w_{21} & z & 0 & & \vdots\\
x^{\binom{3}{2}}y^2 & -w_{31} & 1-w_{31} & z & & \\
\vdots & & & & & \vdots\\
x^{\binom{m}{2}}y^{m-1} & -w_{m1} & \cdots & & \cdots & 1-w_{m1}
\end{pmatrix}.$$

By cofactor expansions (initially along the last row of $\mathbf{B}$), we obtain

$$\det \mathbf{B} = \det \mathbf{N_m} + zqx^my \det \mathbf{N_{m-1}}. \tag{15}$$

And let $\mathbf{C_{m-1}}$ be the $(m-1)\text{x}(m-1)$ matrix obtained by deleting the first row and column of $\mathbf{N_m}$. So, for example,

$$\mathbf{C_4} = \begin{pmatrix} 1-w_{21} & z & 0 & 0 \\ -w_{31} & 1-w_{32} & z & 0 \\ -w_{41} & -w_{42} & 1-w_{43} & z \\ -w_{51} & -w_{52} & -w_{53} & 1-w_{54} \end{pmatrix}.$$

By employing cofactor expansions (also, initially along the last row of $\mathbf{A}$), we see that

$$\det \mathbf{A} = \det \mathbf{C_{m-1}} + zqx^m y \det \mathbf{C_{m-2}}. \tag{16}$$

Again, by employing co-factor expansions along the last row of $\mathbf{C_4}$, we see that

$$\det \mathbf{C_4} = (1-w_{54}) \det \mathbf{C_3} + zw_{53} \det \mathbf{C_2} - w_{52}z^2 \det \mathbf{C_1} + w_{51}z^3 \det \mathbf{C_0},$$

where $\det \mathbf{C_0} := 1$. In general, a cofactor expansion along the last row of $\mathbf{C_m}$ yields for $m \geq 1$

$$\det \mathbf{C_m} = (1-w_{m+1m}) \det \mathbf{C_{m-1}} + \sum_{æ=1}^{m-1} (-1)^{m-1-j} w_{m+1j} z^{m-j} \det \mathbf{C_{j-1}}.$$

Once again making the replacement $w_{ij} = x^{\binom{i}{2}-\binom{j}{2}} y^{i-j}$, we have for $m \geq 1$

$$\det \mathbf{C_m} = (1-x^m y) \det \mathbf{C_{m-1}} + \sum_{æ=1}^{m-1} (-1)^{m-1-j} x^{\binom{m+1}{2}-\binom{j}{2}} y^{m+1-j} z^{m-j} \det \mathbf{C_{j-1}}. \tag{17}$$

Dropping m by 1 and multiplying this equation by $-x^m yz$, we obtain

$$\begin{aligned} &- x^m yz \det \mathbf{C_{m-1}} \\ &= -x^m yz(1-x^{m-1}y) \det \mathbf{C_{m-2}} + \sum_{æ=1}^{m-2} (-1)^{m-1-j} x^{\binom{m+1}{2}-\binom{j}{2}} y^{m+1-j} z^{m-j} \det \mathbf{C_{j-1}}. \end{aligned} \tag{18}$$

By subtracting (18) from (17), we obtain

$$\begin{aligned} &\det \mathbf{C_m} + x^m yz \det \mathbf{C_{m-1}} \\ &\quad = (1-x^m y) \det \mathbf{C_{m-1}} + x^m yz(1-x^{m-1}y) \det \mathbf{C_{m-2}} + x^{2m-1}y^2 z \det \mathbf{C_{m-2}}. \end{aligned}$$

Simplifying,

$$\det \mathbf{C_m} = (1-x^m y(1+z)) \det \mathbf{C_{m-1}} + x^m yz \det \mathbf{C_{m-2}}, \tag{19}$$

where $\det \mathbf{C_{-1}} := 1$; $\det \mathbf{C_0} = 1$; $\det \mathbf{C_1} = 1-xy = 1-w_{21}$.

For ease of notation in the remainder of the paper, we abbreviate $\det \mathbf{C_m}$ as C_m, and define the generating function $C(t) = \sum_{m\geq 0} C_m t^m$. By multiplying equation (19) by t^m and then summing from 1 to infinity, we obtain

$$C(t) - 1 = tC(t) - (1+z)xytC(xt) + x^2yt^2zC(xt) + xyzt.$$

Therefore

$$C(t) = \frac{1+xyzt}{1-t} - xytC(xt)\frac{1+z(1-xt)}{1-t}. \tag{20}$$

Again to simplify the notation, substitute $f(t) := \frac{1+xyzt}{1-t}$ and $\varphi(t) := -xyt\frac{1+z(1-xt)}{1-t}$, and iterate the previous equation to obtain:

$$C(t) = f(t) + \varphi(t)C(xt) = f(t) + \varphi(t)f(xt) + \varphi(t)\varphi(xt)C(x^2t). \tag{21}$$

Repeatedly iterating (assuming $|x| < 1$), we obtain

$$\begin{aligned} C(t) &= \sum_{j\geq 0} f(x^jt)\prod_{\text{ß}=0}^{j-1}\varphi(x^it) \\ &= \sum_{j\geq 0}(-1)^j\frac{1+x^{j+1}yzt}{1-x^jt}x^{\binom{j+1}{2}}y^jt^j\prod_{\text{ß}=0}^{j-1}\frac{1+z(1-x^{i+1}t)}{1-x^it}. \end{aligned}$$

Recall that $z = \frac{-1}{1-x}$ which implies $1+z = \frac{-x}{1-x}$. Therefore,

$$\begin{aligned} C(t) &= \sum_{j\geq 0}(-1)^j(1+x^{j+1}yzt)x^{\binom{j+1}{2}}y^jt^j\frac{\prod_{\text{ß}=1}^{j}(1-\frac{zx^it}{1+z})}{\prod_{\text{ß}=0}^{j}(1-x^it)}(1+z)^j \\ &= \sum_{j\geq 0}(-1)^j(1+x^{j+1}yzt)x^{\binom{j+1}{2}}y^jt^j(\frac{-x}{1-x})^j\frac{\prod_{\text{ß}=0}^{j-1}(1-x^it)}{\prod_{\text{ß}=0}^{j}(1-x^it)} \\ &= \sum_{j\geq 0}\frac{(1+x^{j+1}yzt)x^{\frac{j(j+3)}{2}}y^jt^j}{(1-x)^j(1-x^jt)}. \end{aligned}$$

For further notational simplification, we let

$$f_j = \frac{(1+x^{j+1}yzt)x^{\frac{j(j+3)}{2}}y^jt^j}{(1-x)^j(1-x^jt)}.$$

Finally, substituting for the remaining z as above and using partial fractions

$$\begin{aligned} f_j &= \frac{x^{1+\frac{j(j+3)}{2}}y^{j+1}t^j}{(1-x)^{j+1}} + \frac{x^{\frac{j(j+3)}{2}}y^j(1-x-xy)t^j}{(1-x)^{j+1}(1-x^jt)} \\ &= \frac{x^{1+\frac{j(j+3)}{2}}y^{j+1}t^j}{(1-x)^{j+1}} + \frac{x^{\frac{j(j+3)}{2}}y^j(1-x-xy)t^j}{(1-x)^{j+1}}\sum_{k\geq 0}x^{jk}t^k. \end{aligned}$$

Hence the mth coefficient of $C(t)$ is given by

$$C_m = \frac{x^{\binom{m+2}{2}}y^{m+1}}{(1-x)^{m+1}} + \sum_{j=0}^{m}\frac{x^{\frac{j^2+3j}{2}-j^2+jm}y^j(1-x-xy)}{(1-x)^{j+1}}$$

So, we obtain the following lemma.

Lemma 2.1. *The determinants C_m of the matrices obtained from $\mathbf{N_{m+1}}$ (see equation (2.3)) by deleting its first row and column are given by*

$$C_m = x^{\binom{m+2}{2}}\left(\frac{y}{1-x}\right)^{m+1} + \frac{1-x-xy}{1-x}\sum_{j=0}^{m} x^{(m+1)j-\binom{j}{2}}\left(\frac{y}{1-x}\right)^{j}. \quad (22)$$

For initial cases, we have $\det \mathbf{N_1} = 1$ and $\det \mathbf{N_2} = 1 - xy - zxy$. By a cofactor expansion along the last row, we obtain for $m \geq 2$

$$\begin{aligned}\det \mathbf{N_m} &= (1-x^{m-1}y)\det \mathbf{N_{m-1}}\\ &+ \sum_{æ=1}^{m-2}(-1)^{m-j}x^{\binom{m}{2}-\binom{j}{2}}y^{m-j}z^{m-1-j}\det \mathbf{N_j} + (-1)^{m-1}x^{\binom{m}{2}}y^{m-1}z^{m-1}.\end{aligned} \quad (23)$$

Dropping m by 1 and multiplying this equation by $-x^{m-1}yz$ (a similar process to that used in a previous section), we obtain for $m \geq 3$

$$\begin{aligned}-x^{m-1}yz\det \mathbf{N_{m-1}} &= -x^{m-1}yz(1-x^{m-2}y)\det \mathbf{N_{m-2}}\\ &+ \sum_{æ=1}^{m-3}(-1)^{m-j}x^{\binom{m}{2}-\binom{j}{2}}y^{m-j}z^{m-1-j}\det \mathbf{N_j} + (-1)^{m-1}x^{\binom{m}{2}}y^{m-1}z^{m-1}.\end{aligned} \quad (24)$$

Subtracting (24) from (23), we obtain

$$\begin{aligned}&\det \mathbf{N_m} + x^{m-1}yz\det \mathbf{N_{m-1}}\\ &= (1-x^{m-1}y)\det \mathbf{N_{m-1}} + x^{m-1}yz(1-x^{m-2}y)\det \mathbf{N_{m-2}} + x^{2m-3}y^2z\det \mathbf{N_{m-2}}\\ &= (1-x^{m-1}y)\det \mathbf{N_{m-1}} + x^{m-1}yz\det \mathbf{N_{m-2}}.\end{aligned}$$

Hence for $m \geq 2$,

$$\det \mathbf{N_m} = (1-x^{m-1}y(1+z))\det \mathbf{N_{m-1}} + x^{m-1}yz\det \mathbf{N_{m-2}} \quad (25)$$

with $\det \mathbf{N_0} = 0$ and $\det \mathbf{N_1} = 1$.

For the rest of the paper we simplify matters by abbreviating $N_m := \det \mathbf{N_m}$ and now define the generating function $N(t) = \sum_{m\geq 0} N_m t^m$. By multiplying equation (25) by t^m, summing from 1 to infinity, we obtain

$$N(t) - t = tN(t) - y(1+z)tN(xt) + xyzt^2N(xt)$$

with $N_{-1} := 0$. Hence

$$N(t) = \frac{t}{1-t} + \frac{xyzt^2 - y(1+z)t}{1-t}N(xt). \quad (26)$$

Repeatedly iterating (26) on t (while recalling that $z = \frac{-1}{1-x}$, and assuming $|x| < 1$), we obtain

$$N(t) = \sum_{j\geq 0} \frac{x^j t}{1 - x^j t} \prod_{ß=0}^{j-1} \frac{yx^i t(\frac{-x^{i+1}t}{1-x} + \frac{x}{1-x})}{1 - x^i t}$$
$$= \sum_{j\geq 0} \frac{x^j t}{1 - x^j t} \prod_{ß=0}^{j-1} \frac{yx^i t}{1 - x}$$
$$= \sum_{j\geq 0} \frac{x^{\frac{j^2+3j}{2}} y^j t^{j+1}}{(1 - x^j t)(1 - x)^j}.$$

Thus, we have our final lemma.

Lemma 2.2. *With* $N_m := \det \mathbf{N_m}$ *(see* (2.3)*)*

(27) $$N_m = [t^m]N(t) = \sum_{j=0}^{m-1} x^{mj - \binom{j}{2}} \left(\frac{y}{1-x}\right)^j.$$

2.4. **The generating function** F**.** Finally, apply (15) and (16) to (14). Then, use lemma 2.1 and lemma 2.2, to obtain:

Theorem 2.3. *The generating function* $F = \sum_{a\geq 1; b\geq 1; s\geq 0} n(a, b, s) x^a y^b q^s$ *for the number of staircases* $1^+2^+3^+ \cdots m^+$ *(tracked by the exponent of variable* q*) contained in particular compositions (of* a *with* b *parts) is given by*

(28) $$F = \frac{N_m - \frac{qx^m y}{1-x} N_{m-1}}{(1-q)x^{\binom{m+1}{2}} \left(\frac{y}{1-x}\right)^m + \frac{1-x-xy}{1-x}\left(N_m - \frac{qx^m y}{1-x} N_{m-1}\right)}.$$

For example, Theorem 2.3 with $q = 1$ yields $F_{q=1} = \frac{1-x}{1-x-y}$, which is the generating function for the number of compositions of n with exactly m parts (see [4]).

By differentiating the generating function F with respect to q and then substituting $q = 1$, we obtain

$$\frac{dF}{dq}\Big|_{q=1} = \frac{x^{\binom{m+1}{2}} \left(\frac{y}{1-x}\right)^m}{\frac{(1-x-xy)^2}{(1-x)^2} \left(\sum_{j=0}^{m-1} x^{mj-\binom{j}{2}} \left(\frac{y}{1-x}\right)^j - \sum_{j=1}^{m-1} x^{mj-\binom{j}{2}} \left(\frac{y}{1-x}\right)^j\right)}$$
$$= \frac{x^{\binom{m+1}{2}} y^m}{(1 - x - xy)^2 (1 - x)^{m-2}}$$
$$= \frac{x^{\binom{m+1}{2}}}{(1-x)^m} \sum_{j\geq 0} (j+1) \frac{x^j y^{m+j}}{(1-x)^j}$$

Next, we extract coefficients; firstly of $[y^l]$ to obtain

$$(\ell-m+1)\frac{x^{\ell+\binom{m}{2}}}{(1-x)^{\ell}}=(\ell-m+1)\sum_{j\geq 0}\binom{\ell+j-1}{j}x^{\ell+j+\binom{m}{2}},$$

and then of $[x^n]$ which leads to the following result.

Corollary 2.4. *The total number of staircases $1^+2^+3^+\cdots m^+$ in all compositions of n with exactly ℓ parts is given by*

$$(\ell-m+1)\binom{n-1-\binom{m}{2}}{\ell-1}.$$

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