A q-SYMMETRIC ALGORITHM AND ITS APPLICATIONS TO SOME COMBINATORIAL SEQUENCES

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ABSTRACT. In this paper, we define the q-analogue of the so-called symmetric infinite matrix algorithm. We find an explicit formula for entries in the associated matrix and also for the generating function of the k-th row of this matrix for each fixed k. This helps us to derive analytic and number theoretic identities with respect to the q-harmonic numbers and q-hyperharmonic numbers of Mansour and Shattuck.

1. INTRODUCTION

The Euler-Seidel algorithm is a useful method to study some recurrence relations and combinatorial sequences such as harmonic numbers, hyperharmonic numbers, Lucas numbers and polynomials, hyper-Fibonacci numbers, Bernoulli, Euler and Genocchi polynomials, among others. For more details, see for example [7, 8, 10, 11, 12, 15, 16, 19].

Dil and Mező [9] introduced a new method called the symmetric algorithm, which is an analogue of the Euler-Seidel method. This new method takes two initial sequences as an input, and the output is an infinite matrix. The elements of this matrix are obtained by the recurrence relation

$$a_n^0 = a_n, \quad a_0^n = a^n, \quad (n \geqslant 0)$$
$$a_n^k = a_{n-1}^k + a_n^{k-1}, \quad (n, k \geqslant 1),$$

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that is, the elements are given as the following scheme shows:

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_n^{k-1} & \cdot & \cdot & \cdot \\ & & & & \downarrow & & & \\ \cdot & \cdot & \cdot & a_{n-1}^k \rightarrow & a_n^k & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Recall that the Euler-Seidel matrix [10] is defined by

$$a_n^0 = a_n, \quad (n \geqslant 0)$$
$$a_n^k = a_n^{k-1} + a_{n+1}^{k-1}, \quad (n \geqslant 0, k \geqslant 1).$$

Clarke et al. [5] introduced a q-analogue of the Euler-Seidel matrix and with this they studied the q-analogue of the results of Dumont and Randrianarivory about the combinatorial interpretations of the coefficients of the Euler-Seidel matrix associated to $n!$ [11]. The q-analogue of the Euler-Seidel matrix is defined by the following recurrences:

$$a_n^0(x,q) = a_n(x,q), \quad a_0^n(x,q) = a^n(x,q), \quad (n \geqslant 0)$$
$$a_n^k(x,q) = xq^n a_n^{k-1} + a_{n+1}^{k-1}(x,q), \quad (n,k \geqslant 1).$$

This algorithm was recently generalized by Cetin-Firengiz and Tuglu [3].

Recently, Ramírez and Shattuck [17] introduced the following q-analogue of the symmetric algorithm:

$$a_n^0(u,v,q) = a_n(x,q), \quad a_0^n(u,v,q) = a^n(x,q), \quad (n \geqslant 0)$$
$$a_n^k(u,v,q) = va_{n-1}^k(u,v,q) + uq^{n+2k-1}a_n^{k-1}(u,v,q), \quad (n,k \geqslant 1).$$

In this paper our goal is to introduce a different q-analogue of the symmetric algorithm. Then we use this new method to study the q-hyperharmonic numbers and q-harmonic numbers. Moreover, we give several analytic and number theoretic identities.

2. A q-analogue of the Symmetric Infinite Matrix

Definition 2.1. *Let $(a_n(x,q))_{n\in\mathbb{N}}, (a^n(x,q))_{n\in\mathbb{N}}$ be two real sequences with $a_0(x,q) = a^0(x,q) = a_0^0(x,q)$. We define the elements of the q-symmetric infinite matrix associated with these sequences via the following recursive formulae:*

$$a_n^0(x,q) = a_n(x,q), \quad a_0^n(x,q) = a^n(x,q), \quad (n \geqslant 0) \tag{1}$$

$$a_n^k(x,q) = a_{n-1}^k(x,q) + xq^n a_n^{k-1}(x,q), \quad (n,k \geqslant 1). \tag{2}$$

Note that if $x = 1 = q$ we obtain the matrix of Dil and Mező [9].

We need some notation from q-theory. The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}},$$

where

$$(a;q)_n := \prod_{j=0}^{n-1} (1 - aq^j)$$

stands for the q-Pochhammer symbol.

Another way to write the q-binomial is

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]_q!}{[k]_q! [n-k]_q!},$$

with $[n]_q = 1 + q + \cdots + q^{n-1}$ and $[n]_q! = [1]_q [2]_q \cdots [n]_q$.

With this notation, we can find an expression for an arbitrary entry of the q-symmetric infinite matrix.

Theorem 2.2. *Let $n, k \geq 0$, not both zero. Then the entries of the q-symmetric infinite matrix are given by*

$$a_n^k(x,q) = \sum_{i=1}^{k} \begin{bmatrix} n+k-i-1 \\ n-1 \end{bmatrix} a_0^i(x,q)(qx)^{k-i} + x^k \sum_{s=1}^{n} \begin{bmatrix} n+k-s-1 \\ k-1 \end{bmatrix} a_s^0(x,q) q^{ks}.$$

Proof. We proceed by induction on $s = n + k$. The statement clearly holds when $n = 0$ or $k = 0$ (in particular, when $s = 1$). Suppose that the result holds for all $i \leq s$. We are going to prove it for $s + 1$, where $n, k \geq 1$. We have two cases; if $s + 1 = (n + 1) + k$, then

$$\begin{aligned}
&a_{n+1}^k(x,q) = a_n^k(x,q) + xq^{n+1} a_{n+1}^{k-1}(x,q) \\
&= \sum_{i=1}^{k} \begin{bmatrix} n+k-i-1 \\ n-1 \end{bmatrix} a_0^i(x,q)(qx)^{k-i} + x^k \sum_{s=1}^{n} \begin{bmatrix} n+k-s-1 \\ k-1 \end{bmatrix} a_s^0(x,q) q^{ks} \\
&+ xq^{n+1} \left(\sum_{s=1}^{k-1} \begin{bmatrix} n+k-i-1 \\ n \end{bmatrix} a_0^i(x,q)(qx)^{k-i-1} \right. \\
&\left. + x^{k-1} \sum_{s=1}^{n+1} \begin{bmatrix} n+k-s-1 \\ k-2 \end{bmatrix} a_s^0(x,q) q^{(k-1)s} \right) \\
&= \sum_{i=1}^{k-1} \left(\begin{bmatrix} n+k-i-1 \\ n-1 \end{bmatrix} + q^n \begin{bmatrix} n+k-i-1 \\ n \end{bmatrix} \right) a_0^i(x,q)(qx)^{k-i} + a_0^k(x,q) \\
&+ x^k \sum_{s=1}^{n} \left(\begin{bmatrix} n+k-s-1 \\ k-1 \end{bmatrix} + q^{n-s+1} \begin{bmatrix} n+k-s-1 \\ k-2 \end{bmatrix} \right) a_s^0(x,q) q^{ks} + x^k a_{n+1}^0(x,q) q^{(n+1)k}.
\end{aligned}$$

From the defining recursions

$$\begin{bmatrix} n \\ j \end{bmatrix} = \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} + q^j \begin{bmatrix} n-1 \\ j \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} n \\ j \end{bmatrix} = q^{n-j} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ j \end{bmatrix},$$

we get that

$$\begin{aligned} a_{n+1}^k(x,q) = \sum_{i=1}^{k-1} & \begin{bmatrix} n+k-i \\ n \end{bmatrix} a_0^i(x,q)(qx)^{k-i} + a_0^k(x,q) \\ & + x^k \sum_{s=1}^{n} \begin{bmatrix} n+k-s \\ k-1 \end{bmatrix} a_s^0(x,q) q^{ks} + x^k a_{n+1}^0(x,q) q^{(n+1)k} \\ & = \sum_{i=1}^{k} \begin{bmatrix} n+k-i \\ n \end{bmatrix} a_0^i(x,q)(qx)^{k-i} + x^k \sum_{s=1}^{n+1} \begin{bmatrix} n+k-s \\ k-1 \end{bmatrix} a_s^0(x,q) q^{ks}. \end{aligned}$$

In the other case when $s+1 = n+(k+1)$, the result similarly holds. □

In the theory of infinite symmetric matrices, the form of the generating function of the rows has crucial importance.

Now we introduce the following generating function:

$$a(z) = \sum_{n=1}^{\infty} a_n^0(x,q) z^n.$$

That is, $a(z)$ is the generating function of the input sequence a_n (initial row).

Theorem 2.3. *Let $(a_n(x,q))_{n\in\mathbb{N}}$ and $(a^n(x,q))_{n\in\mathbb{N}}$ be two initial sequences. Then the generating functions of the kth row of the q-symmetric infinite matrix is*

$$A^k(z) = \sum_{n=1}^{\infty} a_n^k(x,q) z^n = \frac{x^k a(q^k z)}{(z;q)_k} + z \sum_{i=1}^{k} \frac{a_0^i(x,q)(qx)^{k-i}}{(z;q)_{k-i+1}}.$$

Proof. From Theorem 2.2 we get that

$$\begin{aligned}
\sum_{n=0}^{\infty} a_{n+1}^{k+1}(x,q)z^n &= \sum_{n=0}^{\infty}\left(\sum_{i=1}^{k+1}\begin{bmatrix} n+k-i+1 \\ n \end{bmatrix} a_0^i(x,q)(qx)^{k+1-i}\right.\\
&\quad \left. +x^{k+1}\sum_{s=1}^{n+1}\begin{bmatrix} n+k-s+1 \\ k \end{bmatrix} a_s^0(x,q)q^{(k+1)s}\right) z^n\\
&= a_0^1(x,q)(qx)^k \sum_{n=0}^{\infty}\begin{bmatrix} n+k \\ n \end{bmatrix} z^n + \sum_{n=0}^{\infty}\sum_{i=1}^{k}\begin{bmatrix} n+k-i \\ n \end{bmatrix} a_0^{i+1}(x,q)(qx)^{k-i}z^n\\
&+ x^{k+1}\sum_{n=0}^{\infty}\sum_{s=0}^{n}\begin{bmatrix} n+k-s \\ k \end{bmatrix} a_{s+1}^0(x,q)q^{(k+1)(s+1)}z^n\\
&= a_0^1(x,q)(qx)^k \sum_{n=0}^{\infty}\begin{bmatrix} n+k \\ n \end{bmatrix} z^n + \sum_{i=1}^{k} a_0^{i+1}(x,q)(qx)^{k-i}\sum_{n=0}^{\infty}\begin{bmatrix} n+k-i \\ n \end{bmatrix} z^n\\
&+ x^{k+1}\sum_{n=0}^{\infty} a_{n+1}^0(x,q)q^{(k+1)(n+1)}z^n \sum_{n=0}^{\infty}\begin{bmatrix} n+k \\ k \end{bmatrix} z^n\\
&= \sum_{n=0}^{\infty}\begin{bmatrix} n+k \\ n \end{bmatrix} z^n\left(a_0^1(x,q)(qx)^k + x^{k+1}\sum_{n=0}^{\infty} a_{n+1}^0(x,q)q^{(k+1)(n+1)}z^n\right)\\
&+ \sum_{i=1}^{k} a_0^{i+1}(x,q)(qx)^{k-i}\sum_{n=0}^{\infty}\begin{bmatrix} n+k-i \\ n \end{bmatrix} z^n.
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n^{k+1}(x,q)z^n &= \sum_{n=0}^{\infty}\begin{bmatrix} n+k \\ n \end{bmatrix} z^n\left(a_0^1(x,q)(qx)^k z + x^{k+1}\sum_{n=0}^{\infty} a_{n+1}^0(x,q)(q^{k+1}z)^{n+1}\right)\\
&+ \sum_{i=1}^{k} a_0^{i+1}(x,q)(qx)^{k-i}z\sum_{n=0}^{\infty}\begin{bmatrix} n+k-i \\ n \end{bmatrix} z^n\\
&= \sum_{n=0}^{\infty}\begin{bmatrix} n+k \\ n \end{bmatrix} z^n\left(a_0^1(x,q)(qx)^k z + x^{k+1}a(q^{k+1}z)\right)\\
&+ \sum_{i=1}^{k} a_0^{i+1}(x,q)(qx)^{k-i}z\sum_{n=0}^{\infty}\begin{bmatrix} n+k-i \\ n \end{bmatrix} z^n\\
&= x^{k+1}a(q^{k+1}z)\sum_{n=0}^{\infty}\begin{bmatrix} n+k \\ n \end{bmatrix} z^n + \sum_{i=0}^{k} a_0^{i+1}(x,q)(qx)^{k-i}z\sum_{n=0}^{\infty}\begin{bmatrix} n+k-i \\ n \end{bmatrix} z^n\\
&= x^{k+1}a(q^{k+1}z)\frac{1}{(z;q)_{k+1}} + \sum_{i=0}^{k} a_0^{i+1}(x,q)(qx)^{k-i}z\frac{1}{(z;q)_{k-i+1}}.
\end{aligned}$$

□

3. Applications

3.1. **q-hyperharmonic numbers.** Mansour and Shattuck [14] introduced the q-hyperharmonic numbers:

$$H_q(n,0) = \frac{1}{q[n]_q},$$
$$H_q(n,r) = \sum_{i=1}^{n} q^i H_q(i,r-1).$$

The hyperharmonic numbers, as referred to by Conway and Guy [6], correspond to the $q=1$ case of $H_q(n,r)$ and have been an object of previous study (see, e.g., [2]).

In [14], the authors gave a combinatorial proof of the following result. Here it will be proven by the q-symmetric algorithm (1).

Theorem 3.1. *If $n \geq 1, k \geq 1$, then*

$$H_q(n,r) = \sum_{j=1}^{n} \begin{bmatrix} n+r-j-1 \\ r-1 \end{bmatrix} \frac{q^{rj-1}}{[j]_q}. \tag{3}$$

Proof. Let $a_n^0(x,q) = \frac{1}{q[n+1]_q}$ and $a_0^n(x,q) = q^{n-1}$ be given for $n \geq 1$. From the q-symmetric algorithm (1) with $x=q$, we obtain the following infinite matrix:

$$\begin{pmatrix} H_q(1,0) & H_q(2,0) & H_q(3,0) & H_q(4,0) & \cdots \\ H_q(1,1) & H_q(2,1) & H_q(3,1) & H_q(4,1) & \cdots \\ H_q(1,2) & H_q(2,2) & H_q(3,2) & H_q(4,2) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Then from Theorem 2.2 we get:

$$\begin{aligned} a_{n+1}^{k+1}(x,q) &= \sum_{i=1}^{k+1} \begin{bmatrix} n+k-i+1 \\ n \end{bmatrix} q^{2k-i+1} + q^{k+1} \sum_{s=1}^{n+1} \begin{bmatrix} n+k-s+1 \\ k \end{bmatrix} \frac{q^{(k+1)s}}{q[s+1]_q} \\ &= q^k \left(\sum_{i=0}^{k} \begin{bmatrix} n+k-i \\ n \end{bmatrix} q^{k-i} + \sum_{s=0}^{n} \begin{bmatrix} n+k-s \\ k \end{bmatrix} \frac{q^{(k+1)(s+1)}}{[s+2]_q} \right) \\ &= q^k \left(\sum_{l=0}^{k} \begin{bmatrix} n+l \\ n \end{bmatrix} q^{l} + \sum_{h=0}^{n} \begin{bmatrix} k+h \\ k \end{bmatrix} \frac{q^{(k+1)(n-h+1)}}{[n-h+2]_q} \right), \end{aligned}$$

where $l = k-i$ and $h = n-s$. From the q-binomial identity (see, e.g., Theorem 3.4 of [1])

$$\begin{bmatrix} n+m+1 \\ m+1 \end{bmatrix} = \sum_{j=0}^{n} \begin{bmatrix} m+j \\ m \end{bmatrix} q^j, \tag{4}$$

we get

$$(5)\quad a_{n+1}^{k+1}(x,q) = q^k \left(\begin{bmatrix} k+n+1 \\ n+1 \end{bmatrix} + \sum_{h=0}^{n} \begin{bmatrix} k+h \\ k \end{bmatrix} \frac{q^{(k+1)(n-h+1)}}{[n-h+2]_q} \right) = q^k \sum_{h=0}^{n+1} \begin{bmatrix} k+h \\ k \end{bmatrix} \frac{q^{(k+1)(n-h+1)}}{[n-h+2]_q}.$$

Therefore

$$(6)\quad a_{n-1}^{k}(x,q) = H_q(n,k) = q^{k-1} \sum_{h=0}^{n-1} \begin{bmatrix} k+h-1 \\ k-1 \end{bmatrix} \frac{q^{k(n-h-1)}}{[n-h]_q} = q^{k-1} \sum_{s=1}^{n} \begin{bmatrix} k+n-s-1 \\ k-1 \end{bmatrix} \frac{q^{k(s-1)}}{[s]_q} = \sum_{s=1}^{n} \begin{bmatrix} k+n-s-1 \\ k-1 \end{bmatrix} \frac{q^{ks-1}}{[s]_q}.$$

This finalizes the proof. □

The following result has already been proven by Mansour and Shattuck in [14] by a different method.

Theorem 3.2. *The generating function of the q-hyperharmonic numbers is*

$$\sum_{n=1}^{\infty} H_q(n,k) z^n = \frac{-\log_q(1-q^k z)}{q(z;q)_k}, \quad k \geq 0,$$

where $-\log_q(1-t) := \sum_{n=1}^{\infty} \frac{t^n}{[n]_q}$ *is the q-logarithm function.*

Proof. Let $a_n^0(x,q) = \frac{1}{q[n+1]_q}$ and $a_0^n(x,q) = q^{n-1}$ be given for $n \geq 1$. From Theorem 2.3 with $x = q$, we obtain

$$A^k(z) = \sum_{n=1}^{\infty} H_q(n+1,k) z^n = \sum_{n=0}^{\infty} H_q(n+1,k) z^n - H_q(1,k) = \sum_{n=1}^{\infty} H_q(n,k) z^{n-1} - q^{k-1},$$

and thus

$$\sum_{n=1}^{\infty} H_q(n,k) z^n = z A^k(z) + q^{k-1} z.$$

On the other hand,

$$A^k(z) = \frac{q^k a(q^k z)}{(z;q)_k} + z \sum_{i=1}^{k} \frac{q^{2k-i-1}}{(z;q)_{k-i+1}}.$$

By using the following equation

$$\begin{aligned} a(q^k z) &= \sum_{n=1}^{\infty} \frac{1}{q[n+1]_q}(q^k z)^n = \frac{1}{q^k z}\sum_{n=1}^{\infty}\frac{1}{q[n]_q}(q^k z)^n - \frac{1}{q} \\ &= \frac{1}{q^k z}\left(\frac{-\log_q(1-q^k z)}{q}\right) - \frac{1}{q} = \frac{-\log_q(1-q^k z)}{q^{k+1}z} - \frac{1}{q}, \end{aligned}$$

we get

$$A^k(z) = \frac{-\log_q(1-q^k z)}{qz(z;q)_k} - \frac{q^{k-1}}{(z;q)_k} + z\sum_{i=1}^{k}\frac{q^{2k-i-1}}{(z;q)_{k-i+1}}.$$

It is not difficult to show that

$$z\sum_{i=1}^{k}\frac{q^{2k-i-1}}{(z;q)_{k-i+1}} = \frac{q^{k-1}}{(z;q)_k} - q^{k-1},$$

from where it comes that

$$\sum_{n=1}^{\infty} H_q(n,k)z^n = \frac{-\log_q(1-q^k z)}{q(z;q)_k} - q^{k-1}z + q^{k-1}z = \frac{-\log_q(1-q^k z)}{q(z;q)_k}.$$

The proof is then complete. □

Corollary 3.3. *The generating function of the hyperharmonic numbers is*

$$\sum_{n=1}^{\infty} H(n,k)z^n = \frac{-\log(1-z)}{(1-z)^k}, \quad k \geq 0.$$

3.2. **Some number theoretical results for the q-harmonic numbers.** Taking a slightly modified version (see, e.g., [18]) of the Mansour-Shattuck q-harmonic number yields some connections to number theory. Namely, let

$$H_{n,q} = \sum_{k=1}^{n}\frac{1}{[k]_q}. \tag{7}$$

Then

Proposition 3.4. *We have*

$$\sum_{n\geq 1} H_{n,q}q^n = \sum_{n\geq 1} d(n)q^n,$$

where $d(n) = \sum_{d|n} 1$ is the divisor function.

Proof. By definition,

$$H_{n,q} = (1-q)\sum_{k=1}^{n}\frac{1}{1-q^k}.$$

Since

$$\sum_{n\geq 1} d(n)q^n = \sum_{n=1}^{\infty}\frac{q^n}{1-q^n},$$

we have

$$\begin{aligned}\sum_{n=1}^{\infty} H_{n,q}q^n &= (1-q)\sum_{n=1}^{\infty} q^n \sum_{k=1}^{n} \frac{1}{1-q^k} = (1-q)\sum_{k=1}^{\infty}\sum_{n=k}^{\infty} \frac{q^n}{1-q^k} \\ &= (1-q)\sum_{k=1}^{\infty} \frac{q^k}{(1-q^k)(1-q)} = \sum_{k=1}^{\infty} \frac{q^k}{(1-q^k)} = \sum_{n\geq 1} d(n)q^n.\end{aligned}$$

□

3.3. **A recursion with respect to** (7)**.** Since the harmonic numbers satisfy the identity

$$H_n = \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k+1}}{k},$$

one might think

$$H_{n,q} = \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} a_k \tag{8}$$

holds for some sequence a_k with the q-binomial coefficients instead of the classical binomial coefficients. This is so, but a_k does not have a simple form. In order to find a_k, we shall need the notion of the q-Seidel matrix of Clarke [5]. Given a sequence a_n, the q-Seidel matrix is associated to the double sequence a_n^k given by the recurrence

$$\begin{aligned} a_n^0 &= a_n \quad (n \geq 0), \\ a_n^k &= q^n a_n^{k-1} + a_{n+1}^{k-1} \quad (n \geq 0, k \geq 1). \end{aligned}$$

In addition, a_n^0 is called the initial sequence and a_0^n the final sequence of the q-Seidel matrix. Then the identity

$$a_0^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} a_k^0 \tag{9}$$

connects the initial and the final sequence.

Define the generating functions of a_n^0 and a_0^n as

$$a(x) = \sum_{n\geq 0} a_n^0 x^n, \quad \overline{a}(x) = \sum_{n\geq 0} a_0^n x^n,$$

and

$$A(x) = \sum_{n\geq 0} a_n^0 \frac{x^n}{[n]_q!}, \quad \overline{A}(x) = \sum_{n\geq 0} a_0^n \frac{x^n}{[n]_q!}.$$

A proposition given in [5] states that these functions are related by the following equations:

$$\overline{a}(x) = \sum_{n\geq 0} a_n^0 \frac{x^n}{(x;q)_{n+1}}, \tag{10}$$

$$\overline{A}(x) = e_q(x)A(x), \tag{11}$$

where

$$e_q(x) = \sum_{n\geq 0} \frac{x^n}{[n]_q!}$$

is the q-analogue of the exponential function [13]. We introduce the notation $\mathrm{Egf}(a_n)$ and $\mathrm{Gf}(a_n)$ for the exponential and ordinary generating function of a_n, respectively.

To reach our aim posed in (8), our approach is as follows. Let the final sequence be $b_n = H_{n,q}$. We determine the initial sequence $a_n^0 = a_n$. Then $\mathrm{Egf}(b_n) \equiv \mathrm{Egf}(H_{n,q}) = e_q \,\mathrm{Egf}(a_n)$. And, to get $\mathrm{Egf}(a_n)$ we determine a_n by using (10) and

$$\sum_{n\geq 1} H_{n,q} x^n = \frac{1-q}{1-x} \sum_{n\geq 1} \frac{x^n}{1-q^n}.$$

Therefore

$$\mathrm{Gf}(b_n) \equiv \mathrm{Gf}(H_{n,q}) = \frac{1-q}{1-x} \sum_{n\geq 1} \frac{x^n}{1-q^n} = \sum_{n\geq 1} a_n \frac{x^n}{(x;q)_{n+1}}. \tag{12}$$

From this equation a_n can be determined. (Note that $a_0 = 0$.)

Proposition 3.5. *We have*

$$H_{n,q} = \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} a_k,$$

where the sequence a_k is determined recursively by

$$\sum_{k=1}^{n} a_k q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} = \frac{1}{[n]_q} = \frac{1-q}{1-q^n} \quad (a_0 := 0).$$

Proof. The denominator of the right hand side of (12) is

$$\frac{1}{(x;q)_{n+1}} = \frac{1}{1-x}\frac{1}{(qx;q)_n} = \frac{1}{1-x}\frac{(q^n qx;q)_\infty}{(qx;q)_\infty}. \tag{13}$$

The q-binomial theorem [13, Section 1.3] states that

$$\frac{(az;q)_\infty}{(z;q)_\infty} = \sum_{k\geq 0} \frac{(a;q)_k}{(q;q)_k} z^k.$$

Applying this to (13),

$$\frac{1}{1-x}\frac{(q^n qx;q)_\infty}{(qx;q)_\infty} = \frac{1}{1-x} \sum_{k\geq 0} \frac{(q^n;q)_k}{(q;q)_k} (qx)^k.$$

Thus (12) becomes

$$(1-q) \sum_{n\geq 1} \frac{x^n}{1-q^n} = \sum_{n\geq 0} a_n x^n \left(\sum_{k\geq 0} \frac{(q^n;q)_k}{(q;q)_k} (qx)^k \right).$$

Let

$$B_{k,n} = \frac{(q^n;q)_k}{(q;q)_k} q^k,$$

for short. Then

$$B_{k,n} = q^k \begin{bmatrix} n+k-1 \\ k \end{bmatrix}$$

for all n and k. Moreover,

$$(1-q)\sum_{n\geq 1}\frac{x^n}{1-q^n} = \sum_{n\geq 0} a_n x^n \left(\sum_{k\geq 0} B_{k,n}x^k\right). \tag{14}$$

If we write the sums term by term, we get

$$a_0(B_{0,0}+B_{1,0}x+B_{2,0}x^2+\cdots)+a_1x^1(B_{0,1}+B_{1,1}x+B_{2,1}x^2+\cdots)+\cdots$$
$$= a_0B_{0,0}+x(a_0B_{1,0}+a_1B_{0,1})+x^2(a_0B_{2,0}+a_1B_{1,1}+a_2B_{0,2})+\cdots$$

Comparing the coefficients here with those on the left hand side of (14), we have

$$\sum_{k=0}^{n} a_k B_{n-k,k} = \frac{1-q}{1-q^n}.$$

Note that – bacause of (12) – a_0 must be zero. Remember also that a_k is the initial sequence of our q-Seidel matrix, so (9) gives

$$H_{n,q} = \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} a_k. \tag{15}$$

This is our proposition. □

Remark. It is worth to present the first terms of the sequence a_n:

$$\begin{aligned}
a_0 &= 0,\\
a_1 &= 1,\\
a_2 &= -\frac{q^2+q-1}{q+1},\\
a_3 &= \frac{q^5+q^4-q^2-q+1}{q^2+q+1},\\
a_4 &= -\frac{q^9+q^8-2q^5+q^2+q-1}{q^3+q^2+q+1},\\
a_5 &= \frac{q^{14}+q^{13}-q^{10}-q^9-q^8+q^7+q^6+q^5-q^2-q+1}{q^4+q^3+q^2+q+1},\\
a_6 &= -\frac{q^{20}+q^{19}-q^{16}-2q^{14}+q^{12}+q^{11}+q^{10}+q^9-2q^7-q^5+q^2+q-1}{q^5+q^4+q^3+q^2+q+1}.
\end{aligned}$$

It would be interesting to give a simple formula for the numerator.

As a consequence of (11) and (15), we have the following connection:

$$\operatorname{Egf}(H_{n,q}) = e_q \operatorname{Egf}(a_n).$$

3.4. **A relation to the q-Stirling numbers.** The q-Stirling numbers of the first kind [4, p. 155] are defined recursively by

$$s_q(n+1,k) = s_q(n,k-1) + [n]_q s_q(n,k), \tag{16}$$

and $s_q(0,0) = 1,\ s_q(n,0) = 0$ when $n > 0$.

Note that

$$H_{n,q} = \frac{1}{[n]_q!} s_q(n+1,2), \tag{17}$$

where $H_{n,q}$ is defined in (7).

To show this, let $H^2_{n,q} = \frac{1}{[n]_q!} s_q(n+1,2)$. Then

$$H^2_{n,q} = \frac{1}{[n]_q!} s_q(n+1,2) = \frac{1}{[n]_q!} s_q(n,1) + \frac{1}{[n-1]_q!} s_q(n,2) = \frac{1}{[n]_q} + H^2_{n-1,q},$$

hence $H^2_{n,q}$ satisfies the same recursion as $H_{n,q}$. Since $H^2_{1,q} = 1 = H_{1,q}$, the two sequences coincide.

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