

ZETA SERIES GENERATING FUNCTION TRANSFORMATIONS RELATED TO GENERALIZED STIRLING NUMBERS AND PARTIAL SUMS OF THE HURWITZ ZETA FUNCTION

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ABSTRACT. We define a generalized class of modified zeta series transformations generating the partial sums of the Hurwitz zeta function and series expansions of the Lerch transcendent function. The new transformation coefficients we define within the article satisfy expansions by generalized harmonic number sequences as the partial sums of the Hurwitz zeta function. These transformation coefficients satisfy many properties which are analogous to known identities and expansions of the Stirling numbers of the first kind and to the known transformation coefficients employed to enumerate variants of the polylogarithm function series. Applications of the new results we prove in the article include new series expansions of the Dirichlet beta function, the Legendre chi function, BBP-type series identities for special constants, alternating and exotic Euler sum variants, alternating zeta functions with powers of quadratic denominators, and particular series defining special cases of the Riemann zeta function constants at the positive integers $s \geq 3$.

1. INTRODUCTION

1.1. Definitions. The *generalized r -order harmonic numbers*, $H_n^{(r)}(\alpha, \beta)$, are defined as the partial sums of the *modified Hurwitz zeta function*, $\zeta(s, \alpha, \beta)$, defined by the series

$$(1) \quad \zeta(s, \alpha, \beta) = \sum_{k \geq 1} \frac{1}{(\alpha k + \beta)^s}.$$

That is, we define these generalized sequences as the sums

$$(2) \quad H_n^{(r)}(\alpha, \beta) = \sum_{1 \leq k \leq n} \frac{1}{(\alpha k + \beta)^r},$$

where the definition of the “ordinary” r -order harmonic numbers, $H_n^{(r)}$, is given by the special cases of (2) where $H_n^{(r)} \equiv H_n^{(r)}(1, 0)$ [11, §6.3]. Additionally, we define the analogous “modified” Lerch transcendent function, $\Phi(z, s, \alpha, \beta) \equiv \alpha^{-s} \cdot \Phi(z, s, \beta/\alpha)$, for

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$|z| < 1$ or when $z \equiv -1$ by the series

$$(3) \quad \Phi(z, s, \alpha, \beta) = \sum_{n \geq 0} \frac{z^n}{(\alpha n + \beta)^s}.$$

We notice the particular important interpretation that the Lerch transcendent function acts as an ordinary generating function that enumerates the generalized harmonic numbers in (2) according to the coefficient identity

$$H_n^{(r)}(\alpha, \beta) = [z^n] \frac{\Phi(z, s, \alpha, \beta) - \beta^{-s}}{1 - z}, \quad |z| < 1 \vee z = -1, \quad n \geq 0.$$

1.2. Approach to generating the modified zeta function series.

1.2.1. *Organization of the new results in the article.* The approach to enumerating the harmonic number sequences and series for special constants within this article begins in Section 2 with a brief overview of the properties of the harmonic number expansions in (2) obtained through the definition of a generalized Stirling number triangle extending the results in [19, 21]. We can also employ transformations of the generating functions of many sequences, though primarily of the geometric series, to enumerate and approximate the generalized harmonic number sequences in (2) which form the partial sums of the modified Hurwitz zeta function in (1).

In Section 3 we continue to relate the identities given for these Stirling number variants given by elementary symmetric polynomials in Section 2 by forming new transformation coefficients which are defined recursively and formally generated by the complete homogeneous symmetric polynomials over the same sequences. In Section 3.3.2, we prove a generalization of our new zeta series generating function transformations which covers (at least formally) a wider range of zeta function cases than those special functions which we restrict our attention to within this article. The last sections of the article are devoted to providing many new examples and special case series applications which showcase our new combinatorially motivated definitions of these classically analytic series and functions. The next subsections of this introduction below compare and contrast existing methods for transforming a zeta-related sequence via its generating function.

1.2.2. *Related integral transformations and formulas.* Given the ordinary generating function, $A(t)$, of the sequence $\langle a_n \rangle_{n \geq 0}$, we can employ a known integral transformation involving $A(t)$ termwise to enumerate the following modified forms of a_n when $r \geq 2$ is integer-valued as in [3]:

$$\sum_{n \geq 0} \frac{a_n}{(n+1)^r} z^n = \frac{(-1)^{r-1}}{(r-1)!} \int_0^1 \log^{r-1}(t) A(tz) dt.$$

Additionally, many zeta function identities correspond to the bilateral series given by Lindelöf in [16, §2] of the form

$$\sum_{n=-\infty}^{\infty} f(n) = -\frac{1}{2\pi i} \oint_{\gamma} \pi \cot(\pi z) f(z) dz,$$

where γ is any closed contour in \mathbb{C} which contains all of the singular points of f in its interior. For integers $\alpha \geq 2$ and $0 \leq \beta < \alpha$, we can similarly transform the ordinary generating function, $A(t)$, through the previous integral transformation and the α^{th} primitive root of unity, $\omega_{\alpha} = \exp(2\pi i/\alpha)$, to reach an integral transformation for the modified Lerch transcendent function in (3) in the form of the next equation [22].

$$\sum_{n \geq 0} \frac{a_{\alpha n + \beta}}{(\alpha n + \beta + 1)^r} z^{\alpha n + \beta} = \frac{(-1)^{r-1}}{\alpha \cdot (r-1)!} \int_0^1 \log^{r-1}(t) \left(\sum_{0 \leq m < \alpha} \omega_{\alpha}^{-m\beta} A(\omega_{\alpha}^m tz) \right) dt$$

1.2.3. *Enumerating the Lerch and Hurwitz zeta functions by generating functions.* In contrast with the approach in the preceding section to enumerating these special zeta functions by their associated integral formulas, we choose a more general and combinatorially motivated method involving derivatives of a sequence’s ordinary generating function. Our new series transformations defined and proved in this article provide new, purely series-based transformations related to a binomial transformation of generating functions and arbitrary harmonic-number-related coefficients defined exactly by symmetric polynomials. In particular, in Section 3.3.2 we prove that for any sequence $\{f_n\}_{n \geq 0}$ whose ordinary generating function, $F(z) \in C^{\infty}([0, \sigma_F))$, is analytic in some disc, we have exact and formally well-defined series transformations of the modified zeta function series associated with some non-zero arithmetic function $g : \mathbb{N} \rightarrow \mathbb{C} \setminus \{0\}$ of the form

$$\sum_{n \geq 0} \frac{f_n}{g(n)^s} z^n = \sum_{j \geq 1} \left\{ \begin{matrix} s+2 \\ j \end{matrix} \right\}_{g^*} \cdot z^j F^{(j)}(z),$$

where here by assumption $s \in \mathbb{Z}^+$ and where the series multiplier coefficients $\left\{ \begin{matrix} s+2 \\ j \end{matrix} \right\}_{g^*}$ are defined precisely by complete homogeneous symmetric polynomials in g . More precisely, we have that [15, §I.2]

$$\left\{ \begin{matrix} s+2 \\ j \end{matrix} \right\}_{g^*} = \sum_{0 \leq i_1 \leq i_2 \leq \dots \leq i_s \leq j} \frac{1}{g(i_1)g(i_2) \cdots g(i_s)} = [z^s] \prod_{i=0}^j \left(1 - \frac{z}{g(i)} \right)^{-1}.$$

In many respects, our new identities are similar to known *Newton series* variants which provide expansions of a sequence and it’s generating function, and a zeta function term $\{g(n)\}_{n \geq 0}$, in terms of the *forward difference operator*, $\Delta^n[f](x_0) := \sum_{1 \leq i \leq n} \binom{n}{i} (-1)^{n-i} f(i+x_0)$, shifted by any integer constant $0 \leq x_0 \leq x$ in the form of

$$\sum_{n \geq 0} \frac{z^n}{f(n)^s} = \sum_{n \geq 0} \sum_{k=0}^n \sum_{i=0}^k \binom{n}{k} \binom{k}{i} \frac{(-1)^{k-i}}{(ai+b)^s} z^n$$

$$= \frac{1}{1-z} \cdot \sum_{n \geq 0} \left[\sum_{i=0}^n \binom{n}{i} \frac{(-1)^i}{f(i)^s} \right] \left(\frac{z}{1-z} \right)^n, \quad b > 0, \frac{b}{a} \neq 0, -1, -2, \dots.$$

While some of the resulting transformed generating function formulas we obtain here as in

$$(1-z) \cdot \Phi(z, s, a) = \sum_{n \geq 0} \left(\frac{-z}{1-z} \right)^n \sum_{k=0}^n \binom{n}{k} \frac{(-1)^{n-k}}{(k+a)^s},$$

cited originally by [20] are not necessarily new in the literature, our approach to these series is much more combinatorial than the traditionally analytic techniques needed to approximate or exactly evaluate integral representations for these functions or to sum variants of their Newton series expansions (*cf.* [12]). In this sense our new results are significant because they provide a distinctly combinatorial twist on many classically treated analytic methods, integrals, and infinite series.

1.3. Examples of the new results. The main focus of this article is on the applications and expansions of generating function transformations introduced in Section 3 that generalize the forms of the coefficients defined in [20]. The generalizations we employ here to transform geometric-series-based generating functions into series in the form of (3) are primarily corollaries to the results in the first article. The next examples illustrate the new series we are able to obtain using these generalized forms of the generating function transformations proved in the reference and using our new harmonic-number-based expansions developed in the sections of this article below.

1.3.1. BBP-Type formulas and identities. Many special constants such as those given as examples in the next sections satisfy series expansions given by *BBP-type formulas* of the form [2]

$$\text{(BBP-Type Series Formula)} \quad P(s, b, m, A) = \sum_{k \geq 0} b^{-k} \sum_{1 \leq j \leq m} \frac{a_j}{(mk+j)^s}$$

where $A = (a_1, a_2, \dots, a_m)$ is a vector of constants and $m, b, s \in \mathbb{Z}^+$. The term *BBP formula* is used to describe the general structure of series expansions in the previous forms which were studied by the authors Bailey, Borwein and Plouffe in 1997 in their search for rapidly convergent series for certain constants, including the next formula for π given by

$$\pi = \sum_{k \geq 0} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right).$$

In this original paper and in [2] algorithms for computing BBP-type formulas for suitable constants using the *PLSQ algorithm* which relates the necessarily integer coefficients in such expansions of a typically polylogarithmically-related series. We remark that the BBP-type formula expansions defined above are equivalent to the class of infinite series whose terms can be written as rational functions of k with respect to an integer base parameter $b \geq 2$, i.e., convergent sums for constants of the form

$\alpha = \sum_{k \geq 0} P(k)/Q(k) \frac{1}{b^k}$ for fixed polynomials P, Q satisfying $\deg(P) < \deg(Q)$ with $Q(k) \neq 0$ for all $k \in \mathbb{N}$. For example, our last formula for π is represented by the equivalent expansion of

$$\pi = \sum_{k \geq 0} \frac{120k^2 + 151k + 47}{16^k (512k^4 + 1024k^3 + 712k^2 + 194k + 15)}.$$

Example 1.1. A pair of particular examples of first-order BBP-type formulas which we list in this section to demonstrate the new forms of the generalized coefficients listed in Table 3 and Table 4 of the article below provide series representations for a real-valued multiple of π and a special expansion of the natural logarithm function [2, §11] [14, §3].

$$\begin{aligned} \frac{4\sqrt{3}\pi}{9} &= \sum_{k \geq 0} \left(-\frac{1}{8}\right)^k \left(\frac{2}{(3k+1)} + \frac{1}{(3k+2)}\right) \\ &= \sum_{j \geq 0} \frac{8}{9^{j+1}} \left(2 \binom{j+\frac{1}{3}}{\frac{1}{3}}^{-1} + \frac{1}{2} \binom{j+\frac{2}{3}}{\frac{2}{3}}^{-1}\right) \\ \text{Log} \left(\frac{n^2 - n + 1}{n^2}\right) &= \sum_{k \geq 0} \left(-\frac{1}{n^3}\right)^{k+1} \left[\frac{n^2}{3k+1} - \frac{n}{3k+2} - \frac{2}{3k+3}\right] \\ &= -\sum_{j \geq 0} \frac{1}{(n^3+1)^{j+1}} \left(n^2 \binom{j+\frac{1}{3}}{\frac{1}{3}}^{-1} - \frac{n}{2} \binom{j+\frac{2}{3}}{\frac{2}{3}}^{-1} - \frac{2}{3(j+1)}\right) \end{aligned}$$

1.3.2. *New series for special zeta functions.* The next two representative examples of special zeta functions serve to demonstrate the style of the new series representations we are able to obtain from the generalized generating function transformations established by this article.

Example 1.2 (Dirichlet’s Beta Function). The Dirichlet beta function, $\beta(s)$, is defined for $\Re(s) > 0$ by the series

$$\beta(s) = \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)^s} = 2^{-s} \Phi(-1, s, 1/2).$$

The series in the previous equation is expanded through the generalized coefficients when $(\alpha, \beta) = (2, 1)$ as in the listings in Table 2. The first few special cases of s over the positive integers are expanded by the following new series [7, cf. §8]:

$$\begin{aligned} \beta(1) &= \sum_{j \geq 0} \frac{1}{2^{j+1}} \binom{j+\frac{1}{2}}{\frac{1}{2}}^{-1} \\ \beta(2) &= \sum_{j \geq 0} \frac{1}{2^{j+1}} \binom{j+\frac{1}{2}}{\frac{1}{2}}^{-1} \cdot \left(1 + H_j^{(1)}(2, 1)\right) \end{aligned}$$

$$\beta(3) = \sum_{j \geq 0} \frac{1}{2^{j+1}} \binom{j + \frac{1}{2}}{\frac{1}{2}}^{-1} \cdot \left(1 + H_j^{(1)}(2, 1) + \frac{1}{2} \left(H_j^{(1)}(2, 1)^2 + H_j^{(2)}(2, 1) \right) \right).$$

Example 1.3 (Legendre's Chi Function). For $|z| < 1$, the Legendre chi function, $\chi_\nu(z)$, is defined by the series

$$\chi_\nu(z) = \sum_{k \geq 0} \frac{z^{2k+1}}{(2k+1)^\nu} = 2^{-\nu} z \times \Phi(z^2, \nu, 1/2).$$

The first few positive integer cases of $\nu \geq 1$ are similarly expanded by the forms of the next series given by

$$\chi_1(z) = \sum_{j \geq 0} \binom{j + \frac{1}{2}}{\frac{1}{2}}^{-1} \frac{z \cdot (-z^2)^j}{(1 - z^2)^{j+1}}$$

$$\chi_2(z) = \sum_{j \geq 0} \binom{j + \frac{1}{2}}{\frac{1}{2}}^{-1} \cdot \left(1 + H_j^{(1)}(2, 1) \right) \frac{z \cdot (-z^2)^j}{(1 - z^2)^{j+1}}$$

$$\chi_3(z) = \sum_{j \geq 0} \binom{j + \frac{1}{2}}{\frac{1}{2}}^{-1} \cdot \left(1 + H_j^{(1)}(2, 1) + \frac{1}{2} \left(H_j^{(1)}(2, 1)^2 + H_j^{(2)}(2, 1) \right) \right) \frac{z \cdot (-z^2)^j}{(1 - z^2)^{j+1}}.$$

Other special function series and motivating examples of these generalized generating function transformations we consider within the examples in Section 4 of the article include special cases of the Hurwitz zeta and Lerch transcendent function series in (1) and (3). In particular, we consider concrete new series expansions of BBP-like series for special constants, *alternating* and *exotic Euler sums* with cubic denominators, *alternating zeta function* sums with quadratic denominators, *polygamma functions*, and several particular series defining the *Riemann zeta function*, $\zeta(2k+1)$, over the odd positive integers.

2. GENERALIZED STIRLING NUMBERS OF THE FIRST KIND

2.1. Definition and generating functions. We first define a generalized set of coefficients in the symbolic polynomial expansions of the next products over x as an extension of the results first given in [19, 21]¹.

$$(4) \quad \left[\begin{matrix} n \\ k \end{matrix} \right]_{(\alpha, \beta)} = [x^{k-1}] x(x + \alpha + \beta)(x + 2\alpha + \beta) \cdots (x + (n-1)\alpha + \beta) [n \geq 1]_\delta$$

The polynomial coefficients of the powers of x in (4) are then defined by the following triangular recurrence for natural numbers $n, k \geq 0$:

$$(5) \quad \left[\begin{matrix} n \\ k \end{matrix} \right]_{(\alpha, \beta)} = (\alpha n + \beta - \alpha) \left[\begin{matrix} n-1 \\ k \end{matrix} \right]_{(\alpha, \beta)} + \left[\begin{matrix} n-1 \\ k-1 \end{matrix} \right]_{(\alpha, \beta)} + [n = k = 0]_\delta.$$

¹ The notation for *Iverson's convention*, $[n = k]_\delta = \delta_{n,k}$, is consistent with its usage in [11].

j \ k	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0
2	0	3	1	0	0	0	0	0	0
3	0	15	8	1	0	0	0	0	0
4	0	105	71	15	1	0	0	0	0
5	0	945	744	206	24	1	0	0	0
6	0	10395	9129	3010	470	35	1	0	0
7	0	135135	129072	48259	9120	925	48	1	0
8	0	2027025	2071215	852957	185059	22995	1645	63	1

TABLE 1. The Generalized Stirling Numbers of the First Kind, $\left[\begin{smallmatrix} k \\ j \end{smallmatrix} \right]_{(2,1)}$

We also easily arrive at generating functions for the column sequences and for the generalized analogs to the *Stirling convolution polynomials*, $\sigma_n(x)$ and $\sigma_n^{(\alpha)}(x)$, defined by

$$\sigma_n^{(\alpha,\beta)}(x) = \left[\begin{matrix} x \\ x-n \end{matrix} \right]_{(\alpha,\beta)} \frac{(x-n-1)!}{x!}.$$

The series enumerating these coefficients are expanded as the following closed-form generating functions [11, §6.2] [13] (see the remark below):

$$(6) \quad \sum_{n \geq 0} \left[\begin{matrix} n \\ k \end{matrix} \right]_{(\alpha,\beta)} \frac{z^n}{n!} = \frac{(1-\alpha z)^{-\beta/\alpha}}{k! \alpha^k} \text{Log}(1-\alpha z)^k$$

$$\sum_{n \geq 0} x \sigma_n^{(\alpha,\beta)}(x) z^n = e^{\beta z} \left(\frac{\alpha z e^{\alpha z}}{e^{\alpha z} - 1} \right)^x.$$

When $(\alpha, \beta) = (1, 0), (\alpha, 1 - \alpha)$ we arrive at the definitions of the respective triangular recurrences defining the *Stirling numbers of the first kind*, $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$, and the generalized α -factorial functions, $n!_{(\alpha)}$, from the references [1, 11, 19]. Table 1 lists the first several rows of the triangle in (5) corresponding to the special case of $(\alpha, \beta) = (2, 1)$ as considered in the special case expansions from [21].

Remark. For fixed x, α, β , we have a known identity for the following exponential generating functions, which then implies the first result in (6) by considering powers of x^k as functions of z where $(x)_n$ denotes the Pochhammer symbol [13] [17, §5.2(iii)]:

$$\sum_{n \geq 0} \left(\frac{x + \beta}{\alpha} \right)_n \frac{(\alpha z)^n}{n!} = (1 - \alpha z)^{-(x+\beta)/\alpha}.$$

We can then apply a double integral transformation for the (ordinary) beta function, $B(a, b)$, in the form of [17, §5.12]

$$B(a, b)^2 = \binom{a+b}{a}^{-2} \cdot \frac{(a+b)^2}{a^2 b^2} = \int_0^\infty \int_0^\infty \frac{(ts)^{a-1}}{[(1+t)(1+s)]^{a+b}} dt ds,$$

for real numbers $a, b > 0$ such that $b \in \mathbb{Q} \setminus \mathbb{Z}$ to these generating functions to obtain partially complete integral representations for the transformed series over the modified coefficients in (10) and (12) through (13) of Section 3.

2.2. Expansions by the generalized harmonic number sequences. We find, as in the references [1, 20], that this generalized form of a Stirling–number–like triangle satisfies a number of analogous harmonic number expansions to the Stirling numbers of the first kind given in terms of the partial sums, $H_n^{(r)}(\alpha, \beta) = \sum_{k=1}^n (\alpha k + \beta)^{-r}$, of the *modified Hurwitz zeta function*, $\zeta(s, \alpha, \beta) = \sum_{n \geq 1} 1/(\alpha n + \beta)^s$. For example, we may expand special case formulas for the triangle columns at $k = 2, 3, 4$ in the following forms²:

$$(7) \quad \begin{aligned} \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_{(\alpha, \beta)} &= n!_{(\alpha, \beta)} \times H_n^{(1)}(\alpha, \beta) \\ \begin{bmatrix} n+1 \\ 3 \end{bmatrix}_{(\alpha, \beta)} &= \frac{n!_{(\alpha, \beta)}}{2} \times \left(\left(H_n^{(1)}(\alpha, \beta) \right)^2 - H_n^{(2)}(\alpha, \beta) \right) \\ \begin{bmatrix} n+1 \\ 4 \end{bmatrix}_{(\alpha, \beta)} &= \frac{n!_{(\alpha, \beta)}}{6} \times \left(\left(H_n^{(1)}(\alpha, \beta) \right)^3 - 3H_n^{(1)}(\alpha, \beta) H_n^{(2)}(\alpha, \beta) + 2H_n^{(3)}(\alpha, \beta) \right). \end{aligned}$$

Similarly, we invert to expand the first few cases of the generalized r -order harmonic numbers through products of the coefficients in (5) as [19, cf. §4.3]

$$\begin{aligned} H_n^{(2)}(\alpha, \beta) &= \frac{1}{\left(n!_{(\alpha, \beta)} \right)^2} \left(\begin{bmatrix} n+1 \\ 2 \end{bmatrix}_{(\alpha, \beta)}^2 - 2 \begin{bmatrix} n+1 \\ 1 \end{bmatrix}_{(\alpha, \beta)} \begin{bmatrix} n+1 \\ 3 \end{bmatrix}_{(\alpha, \beta)} \right) \\ H_n^{(3)}(\alpha, \beta) &= \frac{1}{\left(n!_{(\alpha, \beta)} \right)^3} \left(\begin{bmatrix} n+1 \\ 2 \end{bmatrix}_{(\alpha, \beta)}^3 - 3 \begin{bmatrix} n+1 \\ 1 \end{bmatrix}_{(\alpha, \beta)} \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_{(\alpha, \beta)} \begin{bmatrix} n+1 \\ 3 \end{bmatrix}_{(\alpha, \beta)} \right. \\ &\quad \left. + 3 \begin{bmatrix} n+1 \\ 1 \end{bmatrix}_{(\alpha, \beta)}^2 \begin{bmatrix} n+1 \\ 4 \end{bmatrix}_{(\alpha, \beta)} \right) \\ H_n^{(4)}(\alpha, \beta) &= \frac{1}{\left(n!_{(\alpha, \beta)} \right)^4} \left(\begin{bmatrix} n+1 \\ 2 \end{bmatrix}_{(\alpha, \beta)}^4 - 4 \begin{bmatrix} n+1 \\ 1 \end{bmatrix}_{(\alpha, \beta)} \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_{(\alpha, \beta)}^2 \begin{bmatrix} n+1 \\ 3 \end{bmatrix}_{(\alpha, \beta)} \right. \end{aligned}$$

² We define the shorthand factorial function notation for the products as $n!_{(\alpha, \beta)} = \prod_{j=1}^n (\alpha j + \beta)$.

$$\begin{aligned}
 &+ 2 \begin{bmatrix} n+1 \\ 1 \end{bmatrix}_{(\alpha, \beta)}^2 \begin{bmatrix} n+1 \\ 3 \end{bmatrix}_{(\alpha, \beta)}^2 - 4 \begin{bmatrix} n+1 \\ 1 \end{bmatrix}_{(\alpha, \beta)}^3 \begin{bmatrix} n+1 \\ 5 \end{bmatrix}_{(\alpha, \beta)} \\
 &+ 4 \begin{bmatrix} n+1 \\ 1 \end{bmatrix}_{(\alpha, \beta)}^2 \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_{(\alpha, \beta)} \begin{bmatrix} n+1 \\ 4 \end{bmatrix}_{(\alpha, \beta)} \Big).
 \end{aligned}$$

In general, we can use the *elementary symmetric polynomials* implicit to the product-based definition of these generalized Stirling numbers in (4) to show that [15, cf. §I.2; p. 31]

$$\begin{aligned}
 \begin{bmatrix} n+1 \\ k \end{bmatrix}_{(\alpha, \beta)} &= (-1)^k \cdot n!_{(\alpha, \beta)} \times Y_k \left(-H_n^{(1)}(\alpha, \beta), \dots, (-1)^k H_n^{(k)}(\alpha, \beta) \cdot (k-1)! \right) \\
 H_n^{(k)}(\alpha, \beta) &= (-1)^k (k+1) \begin{bmatrix} n+1 \\ 1 \end{bmatrix}_{(\alpha, \beta)}^{k+1} [t^{k+1}] \text{Log} \left(\begin{bmatrix} n+1 \\ 1 \end{bmatrix}_{(\alpha, \beta)} + \sum_{j \geq 1} \begin{bmatrix} n+1 \\ j+1 \end{bmatrix}_{(\alpha, \beta)} t^j \right),
 \end{aligned}$$

where $Y_n(x_1, x_2, \dots, x_n)$ denotes the *exponential, or complete, Bell polynomial* whose exponential generating function is given by $\Phi(t, 1) \equiv \exp \left(\sum_{j \geq 1} x_j t^j / j! \right)$ [18, §4.1.8].

Additionally, the next recurrences are obtained for the generalized harmonic numbers in terms of these coefficients corresponding to the partial sums in our definition of the “*modified*” Hurwitz zeta function, $\zeta(s, \alpha, \beta) = \alpha^{-s} \times \zeta(s, \beta/\alpha)$.

$$\begin{aligned}
 H_n^{(p)}(\alpha, \beta) &= \sum_{1 \leq j < p} \begin{bmatrix} n+1 \\ p+1-j \end{bmatrix}_{(\alpha, \beta)} \frac{(-1)^{p+1-j} H_n^{(j)}(\alpha, \beta)}{n!_{(\alpha, \beta)}} + \begin{bmatrix} n+1 \\ p+1 \end{bmatrix}_{(\alpha, \beta)} \frac{p(-1)^{p+1}}{n!_{(\alpha, \beta)}} \\
 H_{n+1}^{(p)}(\alpha, \beta) &= H_n^{(p)}(\alpha, \beta) + \begin{bmatrix} n+1 \\ p \end{bmatrix}_{(\alpha, \beta)} \frac{(-1)^{p+1}}{(n+1)!_{(\alpha, \beta)}} \\
 &\quad + \sum_{1 \leq j < p} \begin{bmatrix} n+2 \\ p+1-j \end{bmatrix}_{(\alpha, \beta)} \frac{(-1)^{p+1-j}}{(\alpha n + \alpha + \beta)^j (n+1)!_{(\alpha, \beta)}} \\
 H_{n+1}^{(p)}(\alpha, \beta) &= H_n^{(p)}(\alpha, \beta) + \frac{1}{(\alpha n + \alpha + \beta)^{p-1}} \\
 &\quad + \frac{(-1)^{p-1}}{(n+1)!_{(\alpha, \beta)}} \left(\begin{bmatrix} n+1 \\ p \end{bmatrix}_{(\alpha, \beta)} + \begin{bmatrix} n+1 \\ p-1 \end{bmatrix}_{(\alpha, \beta)} \right) \\
 &\quad + \begin{bmatrix} n+2 \\ p \end{bmatrix}_{(\alpha, \beta)} \frac{(-1)^p}{(\alpha n + \alpha + \beta)(n+1)!_{(\alpha, \beta)}} \\
 &\quad + \sum_{j=0}^{p-3} \begin{bmatrix} n+2 \\ j+2 \end{bmatrix}_{(\alpha, \beta)} \frac{(-1)^{j+1} (\alpha n + \alpha + \beta - 1)}{(\alpha n + \alpha + \beta)^{p-1-j} (n+1)!_{(\alpha, \beta)}}
 \end{aligned}$$

The last equation provides an implicit functional equation between our modified Hurwitz zeta function involving the generalized Stirling numbers of the first kind in (5)

[21]. For $(\alpha, \beta) := (1, 0)$, the previous equation implies new functional equations relating the p -order and $(p - 1)$ -order polylogarithm functions, $\text{Li}_s(z) \equiv \Phi(z, s, 1, 0)$.

3. TRANSFORMATIONS OF ORDINARY POWER SERIES BY GENERALIZED STIRLING NUMBERS OF THE SECOND KIND

3.1. Definitions and preliminary examples. Another approach to the relations of the generalized harmonic number sequences to the forms of the triangles defined by (5) proceeds as in [20]. In particular, the next definitions lead to new expansions of many series and BBP-type formulas for special functions and constants from the introduction and in Section 4 which are implied by the new identities we prove for the series expansions of the *modified Lerch transcendent function*, $\Phi(z, s, \alpha, \beta) = \alpha^{-s} \Phi(z, s, \beta/\alpha)$.

$$(8) \quad \begin{aligned} \left\{ \begin{matrix} k \\ j \end{matrix} \right\}_{(\alpha, \beta)^*} &= \frac{1}{j!} \sum_{0 \leq m \leq j} \binom{j}{m} \frac{(-1)^{j-m}}{(\alpha m + \beta)^{k-2}} \\ \Phi(z, s, \alpha, \beta) &= \beta^{-s} + \sum_{j \geq 0} \left\{ \begin{matrix} s+2 \\ j \end{matrix} \right\}_{(\alpha, \beta)^*} \frac{j! z^j}{(1-z)^{j+1}} \end{aligned}$$

The definition of the generalized *Stirling numbers of the second kind*³ provided by (8), is given recursively by

$$\left\{ \begin{matrix} k \\ j \end{matrix} \right\}_{(\alpha, \beta)^*} = (\alpha j + \beta) \left\{ \begin{matrix} k+1 \\ j \end{matrix} \right\}_{(\alpha, \beta)^*} + \alpha \cdot \left\{ \begin{matrix} k+1 \\ j-1 \end{matrix} \right\}_{(\alpha, \beta)^*}.$$

Example 3.1 (Formal Series Identities). *The definition of the generalized Stirling numbers of the second kind given in both forms above implies the next new truncated partial power series identities for the “modified” Lerch transcendent function over some sequence, $\langle g_n \rangle$, whose ordinary generating function, $G(z)$, has derivatives of all orders, for $\alpha \geq 1$ and $0 \leq \beta < \alpha$, and for any fixed $u \geq 1, u_0 \geq 0$ (cf. [20]).*

$$(9) \quad \begin{aligned} \sum_{n=1}^u \frac{g_n}{(\alpha n + \beta)^k} z^n &= [z^u] \left(\sum_{j=1}^{u+u_0} \left\{ \begin{matrix} k+2 \\ j \end{matrix} \right\}_{(\alpha, \beta)^*} \frac{(wz)^j G^{(j)}(wz)}{(1-w)} \right) \\ \sum_{1 \leq n \leq u} H_n^{(k)}(\alpha, \beta) z^n &= [z^u] \left(\sum_{1 \leq j \leq u+u_0} \left\{ \begin{matrix} k+2 \\ j \end{matrix} \right\}_{(\alpha, \beta)^*} \frac{(wz)^j \cdot j!}{(1-w)(1-wz)^{j+2}} \right) \\ \sum_{1 \leq n \leq u} H_n^{(k)}(\alpha, \beta) \frac{z^n}{n!} &= [z^u] \left(\sum_{1 \leq j \leq u+u_0} \left\{ \begin{matrix} k+2 \\ j \end{matrix} \right\}_{(\alpha, \beta)^*} \frac{(wz)^j \cdot e^{wz} (j+1+wz)}{(j+1)(1-w)} \right) \\ \sum_{1 \leq n \leq u} \left(\sum_{k=1}^n \frac{t^k}{(\alpha k + \beta)^r} \right) z^n &= [z^u] \left(\sum_{1 \leq j \leq u} \left\{ \begin{matrix} r+2 \\ j \end{matrix} \right\}_{(\alpha, \beta)^*} \frac{(twz)^j \cdot j!}{(1-w)(1-wz)(1-twz)^{j+1}} \right) \end{aligned}$$

³ See the conclusions in Section 5.2 for a short discussion of why we consider these transformation coefficients to be generalized Stirling numbers of the second kind.

j \ k	k						
	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$	$\frac{1}{81}$
2	7	1	1	$\frac{7}{15}$	$\frac{41}{225}$	$\frac{223}{3375}$	$\frac{1169}{50625}$
3	1	1	1	$\frac{19}{35}$	$\frac{859}{3675}$	$\frac{34739}{385875}$	$\frac{1323019}{40516875}$
4	1	1	1	$\frac{187}{315}$	$\frac{27161}{99225}$	$\frac{3451843}{31255875}$	$\frac{406586609}{9845600625}$
5	1	1	1	$\frac{437}{693}$	$\frac{735197}{2401245}$	$\frac{1066933061}{8320313925}$	$\frac{1418417467373}{28829887750125}$
6	1	1	1	$\frac{1979}{3003}$	$\frac{45087479}{135270135}$	$\frac{877474863971}{6093243231075}$	$\frac{15505503106933439}{274470141343773375}$
7	1	1	1	$\frac{4387}{6435}$	$\frac{103349119}{289864575}$	$\frac{2065307132299}{13056949780875}$	$\frac{1488524941286431}{23526012115180575}$
8	1	1	1	$\frac{76627}{109395}$	$\frac{31562623583}{83770862175}$	$\frac{10971718559046811}{64148794273438875}$	$\frac{683894055421671560539}{9824580289359984022875}$

TABLE 2. A Table of the Generalized Coefficients $\{j^k\}_{(2,1)^*} \times j!(-1)^{j-1}$

The generalized harmonic number expansions of the coefficients in (8) are considered next in Section 3.2. An even more general proof of the formal power series transformation suggested in the concluding remarks from [20] is given below in Section 3.3. Table 2, Table 3, and Table 4 each provide listings of useful particular special cases of the generalized transformation coefficients, or alternately, *generalized Stirling numbers of the second kind* within the context of this article, corresponding to $(\alpha, \beta) := (2, 1), (3, 1), (3, 2)$, respectively.

3.2. Harmonic number expansions of the generalized transformation coefficients. For integers $\alpha \geq 1$ and $0 \leq \beta < \alpha$, let the first few special cases of the functions, $\tilde{R}_k(\alpha, \beta; j)$, be defined for natural numbers $j \geq 1$ by

$$\tilde{R}_k(\alpha, \beta; j) := \binom{j + \frac{\beta}{\alpha}}{j} \sum_{m=1}^j \binom{j}{m} \frac{(-1)^{m+1}}{(\alpha m + \beta)^k} [k \geq 2]_{\delta} + [k = 1]_{\delta} + [k = 0]_{\delta}.$$

These partial functions are expanded in explicit formulas by the next equations.

$$\begin{aligned}
 (10) \quad & \tilde{R}_1(\alpha, \beta; j) = 1 \\
 & \tilde{R}_2(\alpha, \beta; j) = H_j^{(1)}(\alpha, \beta) \\
 & \tilde{R}_3(\alpha, \beta; j) = \frac{1}{2} \left(H_j^{(1)}(\alpha, \beta)^2 + H_j^{(2)}(\alpha, \beta) \right) \\
 & \tilde{R}_4(\alpha, \beta; j) = \frac{1}{6} \left(H_j^{(1)}(\alpha, \beta)^3 + 3H_j^{(1)}(\alpha, \beta)H_j^{(2)}(\alpha, \beta) + 2H_j^{(3)}(\alpha, \beta) \right) \\
 & \tilde{R}_5(\alpha, \beta; j) = \frac{1}{24} \left(H_j^{(1)}(\alpha, \beta)^4 + 6H_j^{(1)}(\alpha, \beta)^2H_j^{(2)}(\alpha, \beta) + 3H_j^{(2)}(\alpha, \beta)^2 \right)
 \end{aligned}$$

j \ k	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	16	4	1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$	$\frac{1}{256}$
2	-17	1	1	$\frac{5}{14}$	$\frac{41}{392}$	$\frac{311}{10976}$	$\frac{2273}{307328}$
3	1	1	1	$\frac{59}{140}$	$\frac{2671}{19600}$	$\frac{107369}{2744000}$	$\frac{4060291}{384160000}$
4	1	1	1	$\frac{212}{455}$	$\frac{133849}{828100}$	$\frac{73174943}{1507142000}$	$\frac{37005870001}{2742998440000}$
5	1	1	1	$\frac{727}{1456}$	$\frac{1936973}{10599680}$	$\frac{4393719979}{77165670400}$	$\frac{9104269630637}{561766080512000}$
6	1	1	1	$\frac{7271}{13832}$	$\frac{384155263}{1913242240}$	$\frac{17071846526411}{264639666636800}$	$\frac{686298711281124727}{36604958689202176000}$
7	1	1	1	$\frac{23789}{43472}$	$\frac{14322370919}{66143517440}$	$\frac{7187615461845233}{100638684655308800}$	$\frac{3237486239486747349191}{153123771476745445376000}$
8	1	1	1	$\frac{76801}{135850}$	$\frac{238206415289}{1033492460000}$	$\frac{611558324636496331}{7862397238696000000}$	$\frac{1400156984227714635455249}{5981397323310368960000000}$

TABLE 3. A Table of the Generalized Coefficients $\{j^k\}_{(3,1)^*} \times j!(-1)^{j-1}$

j \ k	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	25	5	1	$\frac{1}{5}$	$\frac{1}{25}$	$\frac{1}{125}$	$\frac{1}{625}$
2	-14	2	1	$\frac{11}{40}$	$\frac{103}{1600}$	$\frac{899}{64000}$	$\frac{7567}{2560000}$
3	4	2	1	$\frac{139}{440}$	$\frac{15757}{193600}$	$\frac{1609291}{85184000}$	$\frac{155016733}{37480960000}$
4	4	2	1	$\frac{527}{1540}$	$\frac{446837}{4743200}$	$\frac{334869917}{14609056000}$	$\frac{233183599997}{44995892480000}$
5	4	2	1	$\frac{1889}{5236}$	$\frac{28606807}{274156960}$	$\frac{378441183599}{14354858425600}$	$\frac{4602491925840703}{751620387164416000}$
6	4	2	1	$\frac{19619}{52360}$	$\frac{61764761}{548313920}$	$\frac{4214471373881}{143548584256000}$	$\frac{10491182677877357}{1503240774328832000}$
7	4	2	1	$\frac{66337}{172040}$	$\frac{4956449573}{41436866240}$	$\frac{7985964568560547}{249507946377536000}$	$\frac{466567679887167456041}{60095485932707810816000}$
8	4	2	1	$\frac{110258}{279565}$	$\frac{43971566839}{350141519728}$	$\frac{4710810017671083829}{137042239547861648000}$	$\frac{3642461006944413986125043}{429096793431016946178944000}$

TABLE 4. A Table of the Generalized Coefficients $\{j^k\}_{(3,2)^*} \times j!(-1)^{j-1}$

$$+ 8H_j^{(1)}(\alpha, \beta)H_j^{(3)}(\alpha, \beta) + 6H_j^{(4)}(\alpha, \beta).$$

For larger cases of $k > 5$, we employ the following heuristic to generate the harmonic number expansions of these functions, which for concrete special cases are easily obtained from *Mathematica's* Sigma package:

$$(11) \quad \tilde{R}_m(\alpha, \beta; j) = \sum_{i=0}^{m-2} \frac{\tilde{R}_{m-2-i}(\alpha, \beta; j)}{(m-1)} \cdot H_j^{(i+1)}(\alpha, \beta) + [m=1]_\delta.$$

In general, these harmonic-number-based expansions are enumerated by the generating function products for the *complete homogeneous symmetric functions*, $h_k \equiv h_k(x_1, x_2, \dots, x_n)$, in the special case where $x_j := (\alpha j + \beta)^{-1}$ [15, §I.2]:

$$\begin{aligned} \tilde{R}_m(\alpha, \beta; j) &= [z^m] \left\{ \prod_{i=1}^j \frac{1}{1 - x_i \cdot z} \right\} \\ &= \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq j} \frac{1}{(\alpha i_1 + \beta)(\alpha i_2 + \beta) \cdots (\alpha i_k + \beta)}. \end{aligned}$$

We then define a close analog to the harmonic number expansions in [20] through the expansions of these composite functions as

$$(12) \quad \left\{ \begin{matrix} k+2 \\ j \end{matrix} \right\}_{(\alpha, \beta)^*} = \left\{ \begin{matrix} k+1 \\ j \end{matrix} \right\}_{(\alpha, \beta)^*} \cdot \frac{1}{\beta} + \left(\frac{j + \beta/\alpha}{\beta/\alpha} \right)^{-1} \frac{(-1)^j}{\beta \cdot j!} \cdot \tilde{R}_k(\alpha, \beta; j)$$

$$(13) \quad = \frac{(-1)^{j-1}}{\beta^k \cdot j!} + \sum_{m=0}^{k-1} \left(\frac{j + \beta/\alpha}{\beta/\alpha} \right)^{-1} \frac{(-1)^j}{j!} \cdot \frac{\tilde{R}_{m+1}(\alpha, \beta; j)}{\beta^{k-m}}.$$

Notice that the heuristic we used to generate more involved cases of the expansions for the functions, $\tilde{R}_k(\alpha, \beta; j)$, imply recurrences for the k -order generalized harmonic number sequences in the following forms when $k \geq 3$:

$$\begin{aligned} H_n^{(k)}(\alpha, \beta) &= \frac{H_n^{(k-1)}(\alpha, \beta)}{\beta} + \sum_{j=0}^n \binom{n+1}{j+1} \left(\frac{j + \beta/\alpha}{\beta/\alpha} \right)^{-1} \left(\frac{(-1)^j}{\beta} \cdot \tilde{R}_k(\alpha, \beta; j) \right) \\ H_n^{(k)}(\alpha, \beta) &= \frac{H_n^{(k-2)}(\alpha, \beta)}{\beta^2} \\ &\quad + \sum_{j=0}^n \binom{n+1}{j+1} \left(\frac{j + \beta/\alpha}{\beta/\alpha} \right)^{-1} (-1)^j \cdot \left(\frac{\tilde{R}_k(\alpha, \beta; j)}{\beta} + \frac{\tilde{R}_{k-1}(\alpha, \beta; j)}{\beta^2} \right). \end{aligned}$$

A pair of harmonic and Hurwitz zeta function related identities that follow from the generalized coefficient definitions in (8) are obtained by similar methods from [20, 21] for $n \geq 1$ as follows:

$$\frac{1}{(\alpha n + \beta)^k} = \sum_{0 \leq j \leq n} \binom{n}{j} \left\{ \begin{matrix} k+2 \\ j \end{matrix} \right\}_{(\alpha, \beta)^*} \cdot j!$$

$$\begin{aligned}
&= \sum_{1 \leq m \leq n} \left(\sum_{m \leq j \leq n} \begin{bmatrix} j \\ m \end{bmatrix} \left\{ \begin{matrix} k+2 \\ j \end{matrix} \right\}_{(\alpha, \beta)^*} (-1)^j \right) (-1)^m n^m \\
H_n^{(k)}(\alpha, \beta) &= \sum_{0 \leq j \leq n} \binom{n+1}{j+1} \left\{ \begin{matrix} k+2 \\ j \end{matrix} \right\}_{(\alpha, \beta)^*} \cdot j! \\
&= \sum_{0 \leq p \leq n+1} \left(\sum_{0 \leq j \leq n} \begin{bmatrix} j+1 \\ p \end{bmatrix} \left\{ \begin{matrix} k+2 \\ j \end{matrix} \right\}_{(\alpha, \beta)^*} \frac{(-1)^{j+1}}{(j+1)} \right) (-1)^p \cdot (n+1)^p.
\end{aligned}$$

3.3. Proofs of the zeta series transformations of formal power series.

3.3.1. *Proofs of the Geometric and Exponential Series Transformations.* We claim that for any sequence, $\langle f(n) \rangle$, which is not identically zero for $n \geq 0$, any natural numbers $k \geq 1$, and a sequence, $\langle g_n \rangle$, whose ordinary generating function, $G(z)$, has derivatives of all orders, we have the (formal) series transformation

$$(14) \quad \sum_{n \geq 1} \frac{g_n}{f(n)^k} z^n = \sum_{j \geq 1} \left\{ \begin{matrix} k+2 \\ j \end{matrix} \right\}_{f^*} z^j G^{(j)}(z),$$

where the coefficients implicit to the right-hand-side series are defined by

$$(15) \quad \left\{ \begin{matrix} k+2 \\ j \end{matrix} \right\}_{f^*} = \frac{1}{j!} \sum_{1 \leq m \leq j} \binom{j}{m} \frac{(-1)^{j-m}}{f(m)^k}.$$

As in [20], we primarily only work with these generalized series when the sequence generating function of g_n is some variation of the geometric or exponential series. Therefore, for the content of our article it suffices to prove the next two cases.

Proof of the Geometric Series Case. Let $g_n \equiv 1$ so that its corresponding j^{th} derivative is given by $G^{(j)}(z) = j!/(1-z)^{j+1}$. We proceed to expand the right-hand-side of (14) as follows:

$$\begin{aligned}
\sum_{j \geq 1} \left\{ \begin{matrix} k+2 \\ j \end{matrix} \right\}_{f^*} \frac{j! z^j}{(1-z)^{j+1}} &= \sum_{j \geq 1} \left(\sum_{1 \leq m \leq j} \binom{j}{m} \frac{(-1)^m}{f(m)^k} \right) \frac{(-z)^j}{(1-z)^{j+1}} \\
&= \sum_{m \geq 1} \left(\sum_{j \geq m} \binom{j}{m} \frac{(-z)^j}{(1-z)^{j+1}} \right) \frac{(-1)^m}{m! f(m)^k}.
\end{aligned}$$

For a fixed $c \neq 1$, a known binomial sum identity gives that

$$\sum_{j \geq m} \binom{j}{m} c^j = \frac{c^m}{(1-c)^{m+1}},$$

which then implies that when $c \mapsto -z/(1-z)$ we have

$$\sum_{m \geq 1} \left(\sum_{j \geq m} \binom{j}{m} \frac{(-z)^j}{(1-z)^{j+1}} \right) \frac{(-1)^m}{m! f(m)^k} = \sum_{m \geq 1} (-z)^m \times \frac{(-1)^m}{m! f(m)^k} \quad \square$$

Proof of the Exponential Series Case. Let $g_n \equiv r^n/n!$ so that its corresponding j^{th} derivative is given by $G^{(j)}(z) = r^j e^{rz}$ for all j . In this case, we proceed to expand the right-hand-side of (14) as

$$\begin{aligned} \sum_{j \geq 1} \left\{ \begin{matrix} k+2 \\ j \end{matrix} \right\}_{f^*} (rz)^j e^{rz} &= \sum_{j \geq 1} \left(\sum_{1 \leq m \leq j} \frac{(-1)^m}{m!(j-m)!f(m)^k} \right) (-rz)^j e^{rz} \\ &= \sum_{m \geq 1} \left(\sum_{j \geq m} \frac{(-rz)^j}{(j-m)!} \right) \frac{(-1)^m}{m!f(m)^k} e^{rz} \\ &= \sum_{m \geq 1} (e^{-rz} (-rz)^m) \frac{(-1)^m}{m!f(m)^k} e^{rz} \\ &= \sum_{m \geq 1} \frac{r^m z^m}{m!f(m)^k}. \end{aligned} \quad \square$$

3.3.2. A proof of the generalization to arbitrary zeta function series.

Theorem 3.2 (Generalized Zeta Series Generating Function Expansions). *Fix any sequence $\{f_n\}_{n \geq 0}$ whose ordinary generating function $F(z)$ is analytic on the disc $\{z : |z| < \sigma_F\}$ and suppose that some arithmetic function g is given to be non-zero at all integers $n \geq 0$. Let*

$$Z_{g,s}(z) := \sum_{m=0}^{\infty} \frac{z^m}{g(m)^s},$$

suppose that $\sigma_{g,s}$ denotes the largest radius with respect to z such that the function $Z_{g,s}(z)$ is absolutely convergent on the disk of radius $\sigma_{g,s}$ centered about the origin, and define a non-trivial

$$\sigma_G := \inf_{\{s: \text{Re}(s) > 1 \text{ and } \sigma_{g,s} > 0\}} \sigma_{g,s}.$$

Moreover, we set $\sigma'_F := \min(1, \sigma_F)$ and $\sigma'_G := \min(1, \sigma_G)$ and let the region

$$R_{f,g} := \left\{ z \in \mathbb{C} : \max \left(\frac{\sigma'_F}{1 + \sigma'_F}, \frac{\sigma'_G}{1 + \sigma'_G} \right) \leq |z| < \min \left(\frac{\sigma'_F}{1 - \sigma'_F}, \frac{\sigma'_G}{1 - \sigma'_G} \right) \right\}.$$

Then for all $z \in \Delta_{f,g} \subseteq R_{f,g}$ where $\Delta_{f,g}$ is some non-empty subset of the initial region we have that

$$\sum_{n \geq 0} \frac{f_n z^n}{g(n)^s} = \sum_{j \geq 1} \left\{ \begin{matrix} s+2 \\ j \end{matrix} \right\}_{g^*} z^j \cdot F^{(j)}(z).$$

Proof of the Generalized Series Case. We use the binomial transform applied to the ordinary generating function $F(z)$ in the following form:

(Binomial Transform)
$$\sum_{n \geq 0} \sum_{k=0}^n \binom{n}{k} f_k z^n = \frac{1}{1-z} F \left(-\frac{z}{1-z} \right).$$

For $|z| \in [0, \sigma_F)$ we sum directly by applying the derivative for an analytic function termwise as

$$\begin{aligned}
S_{f,g}(s, z) &:= \sum_{j \geq 1} \left\{ \begin{matrix} s+2 \\ j \end{matrix} \right\}_{g^*} z^j \cdot F^{(j)}(z) \\
&= \sum_{j \geq 1} \left(\sum_{m=1}^j \binom{j}{m} \frac{(-1)^{j-m}}{g(m)^s} \right) \cdot \sum_{n \geq 0} \binom{n}{j} f_n z^n \\
&= \sum_{j \geq 1} \left(\sum_{m \geq 1} \binom{j}{m} \frac{(-1)^{j-m}}{g(m)^s} \right) \cdot [z^j] F \left(-\frac{z}{1-z} \right) \\
&= \sum_{j \geq 1} [z^j] \frac{1}{1+z} Z_{g,s} \left(\frac{z}{1+z} \right) \cdot [z^j] F \left(-\frac{z}{1-z} \right), \quad z \in \Delta_{f,g} \subseteq R_{f,g}.
\end{aligned}$$

Next, we can expand the right-hand-side of the last equation as follows for $z \in R_{f,g}$ such that both functions in this Hadamard product of generating functions are convergent which is guaranteed in some annulus with non-empty interior centered about the origin, a subset in fact, of the region $R_{f,g}$ as in [9, §VI.10.2; pp. 422–427]. Now that we have verified the region of convergence of this modified zeta series, it is simplest to consider the previous equations again from the perspective of performing a binomial transform on the respective generating functions involved. In particular, we have that for all $n \geq 0$

$$\begin{aligned}
[f_n z^n] S_{f,g}(s, z) &= \sum_{j=0}^n \sum_{m=0}^j \binom{n}{j} \binom{j}{m} \frac{(-1)^{j-m}}{g(m)^s} \\
&= [w^n] \left\{ \frac{1}{(1+w)(1-z)} Z_g \left(\frac{z}{1+z} \right) \right\} \Big|_{z \rightarrow w/(1+w)} \\
&= [w^n] Z_g(w) \\
&= \frac{1}{g(n)^s}.
\end{aligned}$$

We remark that the construction above actually works formally with respect to z using operations on formal power series. However, we have carefully defined a region using the triangle inequality for which the transformed zeta series is always an analytic function of $z \in \Delta_{f,g} \subseteq R_{f,g}$. \square

3.3.3. Remarks on symbolic transformation coefficient identities. We observe that most of the identities formulated in [20, §3] are easily restated as symbolic identities for the coefficients in (15). If $f(m)$ is polynomial in m , by expanding $1/f(m)$ in partial fractions over linear factors of m , we arrive at sums over the coefficient forms in (10) of Section 3.2 above. If $1/f(z)$ is a meromorphic function, we may alternately compute the symbolic

coefficients in (15) by the next *Nörlund–Rice integral* over a suitable contour given by [8]

$$\begin{aligned} \left\{ \begin{matrix} k \\ j \end{matrix} \right\}_{f^*} \cdot j! &:= \sum_{1 \leq m \leq j} \binom{j}{m} \frac{(-1)^{j-m}}{f(m)^k} \\ &= \frac{j!}{2\pi i} \oint \frac{f(z)^{-k}}{z(z-1)(z-2)\cdots(z-j)} dz. \end{aligned}$$

We provide a brief overview of examples of several finite sum expansions which are also easily proved along the lines given in the reference, and then quickly move on to the particular cases of the series transformations at hand in this article.

Example 3.3 (Identities for the More General Coefficient Cases). *If we let the r -order f -harmonic numbers, $F_n^{(r)}(f) := \sum_{k=1}^n f(k)^{-r}$, be defined for a non-zero-valued function, $f(n)$, we may expand the coefficients in (15) as (cf. [21])*

$$\begin{aligned} \left\{ \begin{matrix} k \\ j \end{matrix} \right\}_{f^*} &= \sum_{0 \leq i < j} \frac{(j+1)(-1)^{j-1-i}}{(j-1-i)!(i+2)!} \times F_{i+1}^{(k)}(f) \\ &= \sum_{0 \leq i < j} \frac{(j+1)(-1)^{j-1-i}}{(j-1-i)!(i+2)!} \left(F_{i+2}^{(k)}(f) - \frac{1}{f(i+2)^k} \right) \\ &= \sum_{0 \leq i < j} \frac{(-1)^{j-1-i} F_{i+1}^{(k-r)}(f)}{(j-1-i)!(i+2)!} \left(\frac{(i+2)}{f(i+1)^r} + \frac{(j-1-i)}{f(i+2)^r} \right) \\ \left\{ \begin{matrix} k \\ j \end{matrix} \right\}_{f^*} &= \sum_{m=0}^k \sum_{i=1}^k \sum_{r=1}^j \left[\begin{matrix} k \\ i \end{matrix} \right] \left\{ \begin{matrix} i-1 \\ m \end{matrix} \right\} \binom{j}{r} \frac{(-1)^{j-r+m} m! (f(r)-1)^m}{j!(k-1)! f(r)^{m+1}}, \end{aligned}$$

for any integers $j, k \geq 1$ and real-valued $r \in (0, k)$. When $r \notin \mathbb{Z}$, the previous formulas can be used to generalize the explicit harmonic-number-based expansions given in Section 3.2 to non-integral weights of the zeta series parameters.

Notice that we can also similarly define the symbolic generalized Stirling numbers of the first kind by

$$\left[\begin{matrix} n \\ k \end{matrix} \right]_f := [x^k] \prod_j (x + f(j)),$$

and then proceed to derive a whole new related set of even more general symbolic combinatorial identities and properties for these coefficients involving the f -harmonic-numbers, $F_n^{(r)}(f)$.

4. EXAMPLES OF NEW SERIES EXPANSIONS FOR MODIFIED ZETA FUNCTION SERIES

4.1. A special class of alternating Euler sums. A first pair of alternating Euler sums related to the Dirichlet beta function constants are expanded in the following forms [7,

§8] ⁴:

$$\begin{aligned} \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)} H_n &= \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)^2} - \frac{\pi}{2} \text{Log}(2) \\ &= \sum_{j \geq 0} \left(j + \frac{1}{2} \right)^{-1} \frac{(H_j - \text{Log}(2))}{2^{j+1}} \\ \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)^3} H_n &= 3 \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)^4} - \frac{7\pi}{16} \zeta(3) - \frac{\pi^3}{16} \text{Log}(2) \\ &= \sum_{j \geq 0} \left(j + \frac{1}{2} \right)^{-1} \frac{(H_j - \text{Log}(2))}{2^{j+1}} \times \\ &\quad \times \left(1 + H_j^{(1)}(2,1) + \frac{1}{2} \left(H_j^{(1)}(2,1)^2 + H_j^{(2)}(2,1) \right) \right). \end{aligned}$$

The second and fourth series on the right-hand-side of the previous equations follow from an identity for the j^{th} derivatives of the first-order harmonic number generating function given by

$$D_z^{(j)} \left[-\frac{\text{Log}(1-z)}{(1-z)} \right] = \frac{(H_j - \text{Log}(1-z)) j!}{(1-z)^{j+1}},$$

for all integers $j \geq 0$. Since this identity is straightforward to prove by induction, we move quickly along to the next example.

We also observe the key difference between the generalized zeta series transform coefficients introduced in [20] and those defined by (10) and (13) of this article. In particular, the definitions of the generalized coefficients given in this article imply functional equations for a number of special series. For example, the second series in the equations immediately above is given in terms of the first sum in the following form:

$$\begin{aligned} \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)^3} H_n &= \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)^2} H_n \\ &\quad + \sum_{j \geq 0} \left(j + \frac{1}{2} \right)^{-1} \frac{(H_j - \text{Log}(2))}{2^{j+2}} \left(H_j^{(1)}(2,1)^2 + H_j^{(2)}(2,1) \right). \end{aligned}$$

We have similar relations for series defining rational multiples of the *polygamma functions*, $\psi^{s-1}(z/2) - \psi^{s-1}((z+1)/2)$, for example, as in the next pair of related sums

⁴ These special cases of the generalized harmonic number sequences are expanded in terms of the ordinary r -order harmonic numbers, $H_n^{(r)}$, considered in the expansions of [20] as

$$H_n^{(r)}(2,1) = H_{2n+2}^{(r)} - 2^{-r} \cdot H_{n+1}^{(r)} - 1.$$

given by

$$\sum_{k \geq 0} \frac{(-1)^k}{(k+z)^2} = \sum_{j \geq 0} \binom{j+z}{z}^{-1} \left[\frac{1}{z^2} + \frac{1}{z} H_j^{(1)}(1, z) \right] \frac{1}{2^{j+1}}$$

$$\sum_{k \geq 0} \frac{(-1)^k}{(k+z)^3} = \frac{1}{z} \left(\sum_{k \geq 0} \frac{(-1)^k}{(k+z)^2} \right) + \sum_{j \geq 0} \binom{j+z}{z}^{-1} \left[\frac{1}{z} \left(H_j^{(1)}(1, z)^2 + H_j^{(2)}(1, z) \right) \right] \frac{1}{2^{j+2}},$$

which then implies functional equations between the polygamma functions including the following identity:

$$-\frac{z}{4} \left(\psi^{(2)}\left(\frac{z}{2}\right) - \psi^{(2)}\left(\frac{z+1}{2}\right) \right) = \psi'\left(\frac{z}{2}\right) - \psi'\left(\frac{z+1}{2}\right) + 4 \times \sum_{j \geq 0} \binom{j+z}{z}^{-1} \left[H_j^{(1)}(1, z)^2 + H_j^{(2)}(1, z) \right] \frac{1}{2^{j+2}}.$$

4.2. An exotic Euler sum with powers of cubic denominators. Flajolet mentions a more “exotic” family of Euler sums in his article defined for positive integers q by [7]

$$A_q^* = \sum_{n \geq 1} \frac{(-1)^n H_n^2}{[(2n-1)(2n)(2n+1)]^q}.$$

It is not difficult to prove that we have the following two ordinary generating functions for the squares of the first-order harmonic numbers:

$$(16) \quad \sum_{n \geq 0} H_n^2 z^n = \frac{1}{(1-z)} \left(\text{Log}(1-z)^2 + \text{Li}_2(z) \right) = -\frac{1}{(1-z)} \left(2 \text{Li}_2\left(-\frac{z}{1-z}\right) + \text{Li}_2(z) \right).$$

A proof of these two series identities follows from the expansions of the *polylogarithm function*, $\text{Li}_2(z)/(1-z)$, generating the second-order harmonic numbers, $H_n^{(2)}$, in [20, §4]. In particular, we expand the polylogarithm series as

$$\frac{\text{Li}_2(z)}{(1-z)} = - \sum_{j \geq 0} \frac{\left(H_j^2 + H_j^{(2)} \right)}{2(1-z)^2} \left(-\frac{z}{1-z} \right)^j,$$

and then perform the change of variable $z \mapsto -z/(1-z)$ to obtain these results.

Since the derivatives of the polylogarithm functions in each of the equations in (16) are tedious and messy to expand, we do not give any explicit series for these Euler sums, A_q^* . However, we do note that a sufficiently motivated reader may expand these sums by the generalized coefficients we defined in (8) by taking partial fractions of the

denominators of A_q . For example, when $q = 1, 2$ we have the series

$$A_1^* = \sum_{n \geq 1} \frac{(-1)^n H_n^2}{2} \left(\frac{1}{(2n+1)} - \frac{1}{n} + \frac{1}{(2n-1)} \right)$$

$$A_2^* = \sum_{n \geq 1} \frac{(-1)^n H_n^2}{4} \left(\frac{3}{(2n+1)} - \frac{3}{(2n-1)} + \frac{1}{(2n+1)^2} + \frac{1}{n^2} + \frac{1}{(2n-1)^2} \right).$$

To expand the more involved cases of the polylogarithm function derivatives, we first note that for integers $s \geq 2$ and $|z| \leq 1$ we have [10, §2.7]

$$D_z [\text{Li}_s(z)] = \frac{1}{z} \text{Li}_{s-1}(z),$$

where for $r \in \mathbb{Z}^+$ we have the identity that [11, §7.4]

$$\text{Li}_{-r}(z) = \sum_{0 \leq j \leq r} \begin{Bmatrix} r \\ j \end{Bmatrix} \frac{z^j j!}{(1-z)^{j+1}} = \frac{1}{(1-z)^{r+1}} \times \sum_{0 \leq i \leq r} \begin{Bmatrix} r \\ i \end{Bmatrix} z^{i+1},$$

and where the composite derivatives in the second equation of (16) are expanded by *Faà de Bruno's formula* [17, §1.4(iii)]. The two triangles of coefficients in the previous expansions are the *Stirling numbers of the second kind*, $\begin{Bmatrix} n \\ k \end{Bmatrix} = S(n, k) = n! \cdot [z^n] \left\{ \frac{(e^z - 1)^k}{k!} \right\}$, and the *first-order Eulerian numbers*, $\begin{Bmatrix} n \\ k \end{Bmatrix} = n! \cdot [z^n w^k] \left\{ \frac{1-w}{e^{(w-1)z} - w} \right\}$, respectively, for integers $n \geq 0$ and $0 \leq k \leq n$ [11, §6.1–6.2; §7.4].

4.3. Zeta function series with powers of quadratic denominators. We next provide examples of generalized zeta function series over denominator powers of quadratic polynomials. The method employed to expand these particular series is to factor and then take partial fractions to apply the generalized transformation cases we study within this article. For example, we observe that

$$\frac{1}{n^2 + 1} = \frac{i}{2} \left(\frac{1}{n+i} - \frac{1}{n-i} \right)$$

$$\frac{1}{(n^2 + 1)^2} = -\frac{1}{4} \left(\frac{i}{(n-i)} - \frac{i}{(n+i)} + \frac{1}{(n+i)^2} + \frac{1}{(n-i)^2} \right),$$

which immediately leads to the first two of the next series examples.

$$\sum_{n \geq 0} \frac{(-1)^n}{(n^2 + 1)} = \frac{1}{2} (1 + \pi \text{csch}(\pi))$$

$$= \sum_{j \geq 0} \frac{1}{2^{j+2}} \left(\binom{j+i}{i}^{-1} + \binom{j-i}{-i}^{-1} \right)$$

$$\sum_{n \geq 0} \frac{(-1)^n}{(n^2 + 1)^2} = \frac{1}{4} (2 + \pi(1 + \pi \coth(\pi)) \text{csch}(\pi))$$

$$\begin{aligned}
 &= \sum_{j \geq 0} \frac{1}{2^{j+3}} \left(\binom{j+l}{l}^{-1} (2 + lH_j^{(1)}(1, l)) - \binom{j-l}{-l}^{-1} (-2 + lH_j^{(1)}(1, -l)) \right) \\
 \sum_{n \geq 0} \frac{(-1)^n}{(n^2 + 1)^3} &= \frac{1}{32} \left(16 + 6\pi(1 + \pi \coth(\pi)) \operatorname{csch}(\pi) + \pi^3(3 + \cosh(2\pi)) \operatorname{csch}(\pi)^3 \right) \\
 &= \sum_{j \geq 0} \frac{1}{2^{j+5}} \binom{j+l}{l}^{-1} (8 + 5lH_j^{(1)}(1, l) - H_j^{(1)}(1, l)^2 - H_j^{(2)}(1, l)) \\
 &\quad + \sum_{j \geq 0} \frac{1}{2^{j+4}} \binom{j-l}{-l}^{-1} (8 - 5lH_j^{(1)}(1, -l) - H_j^{(1)}(1, -l)^2 - H_j^{(2)}(1, -l))
 \end{aligned}$$

Another quadruple of trigonometric function series providing additional examples of expanding sums with quadratic denominators by partial fractions is given as follows [10, §1.2]:

$$\begin{aligned}
 \frac{\pi}{\sin(\pi x)} &= \frac{1}{x} + \sum_{n \geq 0} \frac{2(-1)^{n+1}}{x^2 - (n+1)^2} \\
 &= \frac{1}{x} + \sum_{j \geq 0} \left[\binom{j+1-x}{1-x}^{-1} \frac{1}{1-x} - \binom{j+1+x}{1+x}^{-1} \frac{1}{1+x} \right] \frac{1}{2^{j+1}} \\
 &= \frac{1}{x} + \frac{\pi^2}{6}x + \frac{7\pi^4}{360}x^3 + \frac{31\pi^6}{15120}x^5 + \frac{127\pi^8}{604800}x^7 + O(x^9) \\
 \frac{1+x \operatorname{csc}(x)}{x} &= \sum_{n \geq 0} \frac{2x(-1)^n}{x^2 - \pi^2 n^2} \\
 &= \sum_{j \geq 0} \left[\binom{j+\frac{x}{\pi}}{\frac{x}{\pi}}^{-1} \frac{1}{x} + \binom{j-\frac{x}{\pi}}{-\frac{x}{\pi}}^{-1} \frac{1}{x} \right] \frac{1}{2^{j+1}} \\
 &= \frac{2}{x} + \frac{x}{6} + \frac{7}{360}x^3 + \frac{31}{15120}x^5 + \frac{127}{604800}x^7 + O(x^9) \\
 \pi \sec(\pi x) &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n+x+\frac{1}{2}} \\
 &= \frac{1}{(x+\frac{1}{2})} - \sum_{j \geq 0} \binom{j+x+\frac{3}{2}}{x+\frac{3}{2}}^{-1} \frac{1}{2^{j+1}(x+\frac{3}{2})} \\
 &\quad + \sum_{j \geq 0} \binom{j+\frac{1}{2}-x}{\frac{1}{2}-x}^{-1} \frac{1}{2^{j+1}(\frac{1}{2}-x)}
 \end{aligned}$$

$$= \pi + \frac{\pi^3}{2}x^2 + \frac{5\pi^5}{24}x^4 + \frac{61\pi^7}{720}x^6 + \frac{277\pi^9}{8064}x^8 + O(x^{10})$$

$$\begin{aligned} \frac{2 + x(1 + x \cot(x)) \csc(x)}{4x^4} &= \sum_{n \geq 0} \frac{(-1)^n}{(x^2 - \pi^2 n^2)^2} \\ &= \sum_{b=\pm 1} \sum_{n \geq 0} \frac{(-1)^n}{4} \left(\frac{b}{x^3(\pi n + bx)} + \frac{1}{x^2(\pi n + bx)^2} \right) \\ &= \sum_{b=\pm 1} \sum_{j \geq 0} \binom{j + \frac{bx}{\pi}}{j}^{-1} \left(\frac{2}{x^4} + \frac{b}{x^3} H_j^{(1)}(\pi, bx) \right) \frac{1}{2^{j+3}} \\ &= \frac{1}{x^4} - \frac{7}{720} - \frac{31x^2}{15120} - \frac{127x^4}{403200} - \frac{73x^6}{1710720} + O(x^7). \end{aligned}$$

Remark (Expansions of General Zeta Series with Quadratic Denominators). *For general quadratic zeta series of the form*

$$(17) \quad \sum_{n \geq 0} \frac{(-1)^n z^n}{(an^2 + bn + c)^{s'}}$$

for integers $s \geq 1$ and constants $a, b, c \in \mathbb{R}$, we can apply the same procedure of factoring the denominators into linear factors of z and taking partial fractions to write the first sums as a finite sum over functions of the form $\Phi(z, s, \alpha, \beta)$ for variable parameters, $\alpha, \beta \in \mathbb{C}$. We also note that a less standard definition of the Lerch transcendent function, $\Phi(z, s, a)$, is given by

$$\Phi^*(z, s, a) = \sum_{n \geq 0} \frac{z^n}{\{(z+a)^2\}^{s/2}} = \sum_{n \geq 0} \frac{z^n}{(z^2 + 2az + a^2)^{s/2}}, \text{ when } \Re(a) > 0,$$

which also suggests an approach to a reduction to non-integer order exponents s from the general quadratic zeta series in the forms of (17). The key distinguishing factor in the less standard definition of the classical special function in the last equation is that it allows us to write $\Phi^*(z, 2s, a)$ in the form of a quadratic zeta series with exponent s as in the more general expansions defined in (17) above. Additionally, when $\Re(a) \leq 0$ and $a \neq 0$ avoids the negative integers, the right-hand-side series in the previous equation actually defines a distinct variant of the classical series in the form of the first equation in (17).

4.4. Special series identities for the Riemann zeta function.

4.4.1. *Series generating the Riemann zeta function at the even integers.* We first consider the following series identity for the zeta function constant, $\zeta(3)$, given by [4, §7.10.2]

$$\zeta(3) = \frac{2\pi^2}{9} \left(\text{Log}(2) + 2 \times \sum_{k \geq 0} \frac{\zeta(2k)}{2^{2k}(2k+3)} \right)$$

We know the next ordinary generating function as [17, §25.8]

$$\sum_{k \geq 0} \zeta(2k)z^k = -\frac{\pi\sqrt{z}}{2} \cot(\pi\sqrt{z}),$$

where $D_z^{(n)}[f(z) \cdot g(z)] = \sum_k \binom{n}{k} f^{(k)}(z)g^{(n-k)}(z)$, $D_z[\cot(z)] = -1 - \cot^2(z)$, and where we can expand the n^{th} derivatives of the cotangent function according to the known formula on the *Wolfram Functions* website as

$$D_z^{(n)}[\cot(z)] = \cot(z) \cdot [n = 0]_\delta - \csc^2(z) \cdot [n = 1]_\delta - n \times \sum_{0 \leq j < k < n} \binom{2k}{j} \binom{n-1}{k} \frac{(-1)^k 2^{n-2k} (k-j)^{n-1}}{(k+1) \cdot \sin^{2k+2}(z)} \sin\left(\frac{n\pi}{2} + 2(k-j)z\right),$$

or equivalently through the series for the *polygamma function*, $\psi_n(z) = \sum_{n \geq 0} (n+z)^{-(n+1)}$, as⁵

$$D_z^{(n)}[\cot(\pi z)] = \frac{1}{\pi} ((-1)^n \psi(1-z) - \psi(z)).$$

Then we may expand variants of the first sum for $\zeta(3)$ related to the zeta function constants, $\zeta(2k+1)$, for integers $k \geq 1$. In particular, we expand the first sum as follows:

$$\zeta(3) = \frac{2\pi^2}{9} \left(\text{Log}(2) + 2 \times \sum_{j \geq 0} \binom{j + \frac{3}{2}}{\frac{3}{2}}^{-1} \frac{\left(-\frac{1}{4}\right)^j}{3j!} D_z^{(j)} \left[-\frac{\pi\sqrt{z}}{2} \cot(\pi\sqrt{z}) \right] \Big|_{z=\frac{1}{4}} \right).$$

⁵ We remark that in general we have the following derivative formulas for all $j \geq 0$ where the coefficients $c_{i,j} \in \mathbb{Z}^+$ for all $0 \leq i \leq j$:

$$D_z^{(j)} [z \cdot F(z)] = j \cdot F^{(j-1)}(z) + z \cdot F^{(j)}(z)$$

$$D_z^{(j)} [\sqrt{z} \cdot F(\sqrt{z})] = \frac{1}{2^j \cdot (\sqrt{z})^{2j-1}} \sum_{i=0}^j (-1)^{j-1-i} c_{j,i} (\sqrt{z})^i F^{(i)}(\sqrt{z}).$$

In this case, the expansions of the polygamma function in terms of our new series identities from Section 4.1 provide the relevant derivatives of $F(z) = \cot(\pi z)$ in the series involved in the formulas given below. Additionally, we can apply the first of the previous two formulas in tandem with the next two identities from [6, §4] to expand our formulas exactly by the Hurwitz zeta function when $n \geq 1$ (*cf.* Section 4.5.3):

$$\frac{\pi}{(2n)!} D_x^{(2n)} [\cot(\pi x)] = \zeta(2n+1, x) - \zeta(2n+1, 1-x)$$

$$-\frac{\pi}{(2n-1)!} D_x^{(2n-1)} [\cot(\pi x)] = \zeta(2n, x) + \zeta(2n, 1-x).$$

Related expansions of other series for the zeta function constants over the odd positive integers $s \geq 3$ include the next identity for integers $n \geq 1$ [4, §7.10.2].

$$\zeta(2n+1) = \frac{2(-1)^n (2\pi)^{2n}}{(2n-1)2^{2n}+1} \left(\sum_{1 \leq k < n} \frac{(-1)^{k-1} k}{(2n-2k+1)!} \frac{\zeta(2k+1)}{\pi^{2k}} + \sum_{k \geq 0} \frac{(2k)!}{(2n+2k+1)!} \frac{\zeta(2k)}{2^{2k}} \right)$$

We then arrive at a BBP-type series for the constant, $\zeta(5)$, in the following form when $(a_1, a_2, a_3, a_4, a_5) = \left(\frac{1}{24}, -\frac{1}{6}, \frac{1}{4}, -\frac{1}{6}, \frac{1}{24}\right)$:

$$\zeta(5) = \frac{16\pi^2}{147}\zeta(3) + \frac{32\pi^4}{49} \times \sum_{j \geq 0} \sum_{1 \leq i \leq 5} \binom{j + \frac{i}{2}}{\frac{i}{2}}^{-1} \frac{a_i \left(-\frac{1}{4}\right)^j}{i j!} D_z^{(j)} \left[-\frac{\pi\sqrt{z}}{2} \cot(\pi\sqrt{z}) \right] \Big|_{z=\frac{1}{4}}$$

Suppose that the ordinary generating function for the sequence, $\langle c_k^* \rangle$, is denoted by $C(z)$. Then provided that the function $C(z)$ has j^{th} derivatives with respect to z for $1 \leq j \leq 5$, we can transform this generating function into a generating function enumerating the next sequence enumerated in the form of

$$\sum_{k \geq 0} \left(32k^5 + 240k^4 + 680k^3 + 900k^2 + 548k + 120 \right) c_k^* z^k = 120C(z) + 2400zC'(z) + 5100z^2C''(z) + 2920z^3C^{(3)}(z) + 560z^4C^{(4)}(z) + 32z^5C^{(5)}(z),$$

which implies that

$$\begin{aligned} \tilde{C}_{2k}(z) &:= \sum_{k \geq 0} (2k+1)(2k+2)(2k+3)(2k+4)(2k+5)\zeta(2k)z^k \\ &= 8\pi^6 z^3 \csc^6(\pi\sqrt{z}) + 4\pi^4 z^2 (11\pi^2 z \cot^2(\pi\sqrt{z}) - 60\pi\sqrt{z} \cot(\pi\sqrt{z}) + 75) \csc^4(\pi\sqrt{z}) \\ &\quad + 4\pi^2 z (-30\pi^3 z^{3/2} \cot^3(\pi\sqrt{z}) + 2\pi^4 z^2 \cot^4(\pi\sqrt{z}) + 150\pi^2 z \cot^2(\pi\sqrt{z}) - 300\pi\sqrt{z} \cot(\pi\sqrt{z}) + 225) \csc^2(\pi\sqrt{z}) \\ &\quad - 360\pi\sqrt{z} \cot(\pi\sqrt{z}). \end{aligned}$$

From the generating function expansion in the previous equation, we have another “coerced” variant of a degree-2 BBP-type formula of the following form when the corresponding coefficient sets are defined to be $(b_1, b_2, b_3, b_4, b_5) = \frac{1}{3456} (-25, -160, 0, 160, 25)$ and $(c_1, c_2, c_3, c_4, c_5) = \frac{1}{3456} (6, 96, 216, 96, 6)$:

$$\begin{aligned} \zeta(5) &= \frac{16\pi^2}{147}\zeta(3) \\ &\quad + \frac{32\pi^4}{49} \times \sum_{j \geq 0} \sum_{1 \leq i \leq 5} \left[\binom{j + \frac{i}{2}}{\frac{i}{2}}^{-1} \left(\frac{b_i}{i} + \frac{c_i}{i^2} + \frac{c_i}{i} H_j^{(1)}(2, i) \right) \right] \frac{\left(-\frac{1}{4}\right)^j}{j!} D_z^{(j)} \left[\tilde{C}_{2k}(z) \right] \Big|_{z=\frac{1}{4}}. \end{aligned}$$

4.4.2. *Zeta function constants defined by the alternating Hurwitz zeta function.* We conclude the applications in this section with several series for the *alternating Hurwitz zeta function*, $\zeta^*(s, \alpha, \beta) + \beta^{-s} = \Phi(-1, s, \alpha, \beta)$. In particular, we see that [17, §25.11(x)]

$$\begin{aligned} \sum_{n \geq 0} (-1)^n \left[\frac{1}{(3n+1)^s} - \frac{1}{(3n+2)^s} \right] &= 6^{-s} \left[\zeta\left(s, \frac{1}{6}\right) - \zeta\left(s, \frac{2}{3}\right) + \zeta\left(s, \frac{5}{6}\right) - \zeta\left(s, \frac{1}{3}\right) \right] \\ &= 6^{-s} (2^s - 2) (3^s - 1) \cdot \zeta(s). \end{aligned}$$

Examples of the last series identity for the Riemann zeta function, $\zeta(s)$, when $s = 2, 3, 4$ are given in the following equations in terms of the special cases of the transformation coefficients expanded in Table 3 and Table 4:

$$\begin{aligned} \frac{2\pi^2}{27} &= \sum_{i=1,2} \sum_{j \geq 0} \binom{j + \frac{i}{3}}{\frac{i}{3}}^{-1} \left[\frac{1}{i^2} + \frac{1}{i} H_j^{(1)}(3, i) \right] \frac{(-1)^{i+1}}{2^{j+1}} \\ \frac{13}{18} \zeta(3) &= \sum_{i=1,2} \sum_{j \geq 0} \binom{j + \frac{i}{3}}{\frac{i}{3}}^{-1} \left[\frac{1}{i^3} + \frac{1}{i^2} H_j^{(1)}(3, i) + \frac{1}{2i} \left(H_j^{(1)}(3, i)^2 + H_j^{(2)}(3, i) \right) \right] \frac{(-1)^{i+1}}{2^{j+1}} \\ \frac{7\pi^4}{729} &= \sum_{i=1,2} \sum_{j \geq 0} \binom{j + \frac{i}{3}}{\frac{i}{3}}^{-1} \left[\frac{1}{i^4} + \frac{1}{i^3} H_j^{(1)}(3, i) + \frac{1}{2i^2} \left(H_j^{(1)}(3, i)^2 + H_j^{(2)}(3, i) \right) \right. \\ &\quad \left. + \frac{1}{6i} \left(H_j^{(1)}(3, i)^3 + 2H_j^{(1)}(3, i) H_j^{(2)}(3, i) + 3H_j^{(3)}(3, i) \right) \right] \frac{(-1)^{i+1}}{2^{j+1}}. \end{aligned}$$

4.5. More examples of the series transformations and applications to other special functions.

4.5.1. *Examples of geometric-series-based generating function variants.* Another example of a geometric-series-based zeta series variant is given by

$$\begin{aligned} \tan^{-1}(x) &= \sum_{n \geq 0} \frac{(-1)^n F_{2n+1} t^{2n+1}}{5^n (2n+1)} \\ &= \frac{\sqrt{5}}{2t} \times \sum_{b=\pm 1} \sum_{j \geq 0} \binom{j + \frac{1}{2}}{j}^{-1} \frac{b}{\sqrt{5}} \left[\frac{\left(\frac{b t \varphi t}{\sqrt{5}} \right)^j}{\left(1 - \frac{b t \varphi t}{\sqrt{5}} \right)^{j+1}} - \frac{\left(\frac{b t \Phi t}{\sqrt{5}} \right)^j}{\left(1 + \frac{b t \Phi t}{\sqrt{5}} \right)^{j+1}} \right], \end{aligned}$$

for $t \equiv 2x / \left(1 + \sqrt{1 + \frac{4}{5}x^2} \right)$, where F_{2n+1} denotes the $(2n+1)^{th}$ Fibonacci number whose generating function is expanded in partial fractions as follows for $\varphi, \Phi := \frac{1}{2} (1 \pm \sqrt{5})$ and the real-valued constants $c_1 := 1/\sqrt{5}$ and $c_2 := -1/\sqrt{5}$:

$$\sum_{n \geq 0} F_{2n+1} z^{2n+1} = \frac{1}{2} \cdot \left(\frac{c_1}{1 - \varphi z} - \frac{c_1}{1 + \varphi z} - \frac{c_2}{1 + \Phi z} + \frac{c_2}{1 - \Phi z} \right).$$

We can also form yet other variants of the geometric-series-based transformations by considering Fourier series for special polynomials, such as the *periodic Bernoulli polynomials*, $\widetilde{B}_n(x) \equiv B_n(x - \{x\})$, and the *Euler polynomials*, $E_n(x) = n! \cdot [t^n] 2e^{tx} / (e^t + 1)$, given in the forms of the following particular series expansions for $n \geq 1$ [17, §24.8(i)]:

$$\begin{aligned} \frac{E_{2n-1}(x)}{(2n-1)!} &= \frac{4(-1)^n}{\pi^{2n}} \times \sum_{k \geq 0} \frac{\cos((2k+1)\pi x)}{(2k+1)^{2n}} \\ \frac{E_{2n}(x)}{(2n)!} &= \frac{4(-1)^n}{\pi^{2n+1}} \times \sum_{k \geq 0} \frac{\sin((2k+1)\pi x)}{(2k+1)^{2n+1}}. \end{aligned}$$

Section 4.5.3 provides additional identities for these trigonometric series expanded by the alternating Hurwitz zeta function, $\zeta^*(s, a)$, and its new series expansions given in Section 4.4.2 from above.

4.5.2. *Series involving reciprocals of binomial coefficients.* We can extend the method employed in constructing the results in Section 4.4 to further zeta series enumerating reciprocals of the *central binomial coefficients*, $\binom{2n}{n}$, by first noticing that we have the following generating function:

$$\sum_{n \geq 0} \frac{z^{2n}}{\binom{2n}{n}} = 4 \left(\frac{1}{4-z^2} + \frac{z}{(4-z^2)^{3/2}} \sin^{-1} \left(\frac{z}{2} \right) \right), \quad |z| < 2.$$

Though a formula for the j^{th} derivatives of the right-hand-side function in the last equation is not clear, we may proceed to formulate the corresponding modified zeta series transformations involving the power series expansions of this ordinary generating function. Examples of applications of expanding series of this type through the new generating function transformations include the following three series [17, §25.6(iii)]:

$$\begin{aligned} \zeta(3) &= \frac{5}{2} \times \sum_{k \geq 1} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}} \\ \zeta(2) - \operatorname{csch}^{-1}(2) \sinh^{-1}(2) &= \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)^2} \binom{2n}{n}^{-1} \\ \left(\sin^{-1}(x) \right)^2 &= \frac{1}{2} \times \sum_{n \geq 0} \frac{(2x)^{2n}}{n^2 \binom{2n}{n}}. \end{aligned}$$

4.5.3. *Some new identities for the alternating Hurwitz zeta function.* We are restricted in our new geometric-series-based transformation results by the fact that we cannot set $(1-z)^{-1}$ nor its j^{th} derivatives to have an input of $z = 1$. This is still not particularly restrictive in identifying new series expansions for the Hurwitz zeta function, $\zeta(s, z) = \Phi(1, s, z)$, which in fact converges whenever $\Re(s) > 1$ and $a \notin \mathbb{Z}^- \cup \{0\}$. In particular, in analogous form to the *alternating zeta function* defined in [20], we define the so-termed

alternating Hurwitz zeta function by the series

$$\zeta^*(s, a) := \sum_{n \geq 0} \frac{(-1)^n}{(n+a)^s}, \Re(s) \geq 1, a \neq 0, -1, -2, \dots$$

We prove the next lemma expanding the classical $\zeta(s, a)$ in terms of its alternating series forms.

Lemma 4.1. *For any $s \in \mathbb{C}$ with $\Re(s) > 1$ and any real $a \notin \mathbb{Z}$, we have the following formula:*

$$\zeta(s, a) = \frac{(-1)^s}{2^{1-s} - 1} \cdot \zeta^*(s, -a/2) - \zeta^*(s, a).$$

Proof. First, notice that by direct expansion of the series for $\zeta^*(s, a)$ we have that

$$\begin{aligned} \zeta^*(s, a) &= \sum_{n \geq 0} \frac{1}{(2n+a)^s} - \sum_{n \geq 0} \frac{1}{(2n+1+a)^s} \\ &= 2^{-s} \left[\zeta\left(s, \frac{a}{2}\right) - \zeta\left(s, \frac{a+1}{2}\right) \right]. \end{aligned}$$

Then by applying the multiplication formula for the Hurwitz zeta function we obtain that

$$(i) \quad \zeta^*(s, a) = 2^{1-s} \zeta\left(s, \frac{a}{2}\right) - \zeta(s, a).$$

Next, we need to express the half-argument term in the previous expansion in terms of the alternating Hurwitz zeta function. We do this by expanding

$$\begin{aligned} 2^{1-s} \cdot \zeta\left(s, \frac{a}{2}\right) &= \sum_{n \geq 0} \left[\frac{1}{(n+a/2)^s} + \frac{(-1)^{s+n}}{(n-a/2)^s} \right] \\ &= \zeta(s, a/2) + (-1)^s \zeta^*(s, -a/2), \end{aligned}$$

which then implies that

$$\zeta\left(s, \frac{a}{2}\right) = \frac{(-1)^s}{(2^{1-s} - 1)} \zeta^*(s, -a/2).$$

Finally, we combine the previous equation with (i) above to arrive at our claimed formula. □

Remark (Corollaries). *One immediate relevant corollary is that the Dirichlet L-functions, $L(s, \chi)$, for χ any character modulo $k \geq 2$ are expanded as*

$$\begin{aligned} L(s, \chi) &= \frac{1}{k^s} \sum_{n=1}^k \chi(n) \zeta\left(s, \frac{n}{k}\right) \\ &= \frac{1}{k^s} \sum_{n=1}^k \chi(n) \left[\frac{(-1)^s}{2^{1-s} - 1} \zeta^*\left(s, -\frac{n}{2k}\right) - \zeta^*\left(s, \frac{n}{k}\right) \right]. \end{aligned}$$

This corollary then provides new series expansions of these L-function cases through our new identities for the Lerch functions $\Phi(-1, s, a)$ and $\Phi^(-1, s, a)$ which we have already obtained*

in more generality in the previous subsections of this article. We also have corresponding new expansions of the trigonometric series defined by the next equations for $\nu \in \mathbb{R}_{>1}$ and any fixed $\alpha \in \mathbb{C}$ [5, 6].

$$\begin{aligned} C_\nu(\alpha) &:= \sum_{k \geq 0} \frac{\cos[(2n+1)\alpha]}{(2n+1)^\nu} &&= \Re[\chi_\nu(e^{i\alpha})] \\ S_\nu(\alpha) &:= \sum_{k \geq 0} \frac{\sin[(2n+1)\alpha]}{(2n+1)^\nu} &&= \Im[\chi_\nu(e^{i\alpha})] \end{aligned}$$

In particular, for integers $p, q \geq 1$ such that $\gcd(p, q) = 1$ the following formulas are proved in the references:

$$\begin{aligned} C_\nu\left(\frac{\pi p}{q}\right) &= \frac{1}{(2q)^\nu} \sum_{s=1}^q \zeta\left(\nu, \frac{2s-1}{2q}\right) \cos\left[\frac{(2s-1)\pi p}{q}\right] \\ S_\nu\left(\frac{\pi p}{q}\right) &= \frac{1}{(2q)^\nu} \sum_{s=1}^q \zeta\left(\nu, \frac{2s-1}{2q}\right) \sin\left[\frac{(2s-1)\pi p}{q}\right]. \end{aligned}$$

4.5.4. *Miscellaneous other examples and applications.* One last class of applications of the modified zeta series transformations that is important to mention comprises geometric and exponential-series-based generating functions for which the parameters (α, β) in the coefficients from (8) correspond to the expansion variable in a power series for a special function. We again demonstrate by example [17, cf. §5.9(i)]:

$$\begin{aligned} \Gamma(z) &= \sum_{n \geq 0} \frac{(-1)^n}{(n+z) \cdot n!} + \int_1^\infty t^{z-1} e^{-t} dt \\ &= \sum_{j \geq 0} \frac{e^{-1}}{z} \binom{j+z}{z}^{-1} + \int_1^\infty t^{z-1} e^{-t} dt \\ \sum_{k \geq 1} \frac{z}{(kz+1)^2} &= \sum_{k \geq 0} B_k z^k = \int_0^\infty \frac{tze^{-t}}{e^{tz} - 1} dt \\ &= \sum_{k \geq 1} \frac{(-1)^k z}{(kz+1)^2} + 2 \times \sum_{k \geq 0} \frac{z}{((2k+1)z+1)^2} \\ &= \sum_{k \geq 1} \frac{(-1)^k z}{(kz+1)^2} + 2 \times \sum_{k \geq 0} \frac{(-1)^k z}{((2k+1)z+1)^2} + 4 \times \sum_{k \geq 0} \frac{z}{((4k+3)z+1)^2} \\ &= \sum_{i \geq 0} \sum_{k \geq 0} \frac{2^i (-1)^k z}{((2^i k + 2^i - 1)z + 1)^2}. \end{aligned}$$

5. CONCLUSIONS AND FINAL EXAMPLES

5.1. **Summary.** We have defined and proved special cases of a generalized generating function transform generating modified zeta functions and special zeta series. In Section 2 we connected the harmonic number expansions of generalized Stirling numbers of the first kind to partial sums of the modified Hurwitz zeta function defined by (1) and (3) of Section 1. The primary source of our new examples and applications is the generalization, or at least significant corollary, to the generating function transformations proved in [20]. Section 1.3 of the introduction and Section 4 suggest many more important applications of these generalized forms of the generating function transformations explored in the first article.

5.2. **Relations to variations of the Stirling numbers of the second kind.** The second geometric series transformation identity stated in (8) effectively provides a combinatorial motivation for the known series for the Lerch transcendent function given by (See [20])

$$\Phi(z, s, \alpha, \beta) = \sum_{k \geq 0} \left(\frac{-z}{1-z} \right)^{k+1} \sum_{0 \leq m \leq k} \binom{k}{m} \frac{(-1)^{m+1}}{(\alpha m + \alpha + \beta)^s}.$$

The even more general forms of the transformation coefficients defined by (8) are considered to be Stirling numbers of the second kind “in reverse” in the sense that we have another related generalized form of the generating function transformation motivating the explorations in the first article defined by [11, cf. §7.4] (cf. [21])

$$\left\{ \begin{matrix} k \\ j \end{matrix} \right\}_{\alpha, \beta} := \sum_{0 \leq m \leq j} \binom{j}{m} \frac{(-1)^{j-m} (\alpha m + \beta)^k}{j!} \implies$$

$$\sum_{n \geq 0} (\alpha n + \beta)^k z^n = \sum_{0 \leq j \leq k} \left\{ \begin{matrix} k \\ j \end{matrix} \right\}_{\alpha, \beta} \frac{z^j \cdot j!}{(1-z)^{j+1}}.$$

Moreover, we can extend this analog by observing that we also have the following negative-order identity involving the generalized Stirling numbers of the second kind defined by the last power series transformation identity [17, cf. §26.8(v)]:

$$\sum_{0 \leq j \leq n} (\alpha j + \beta)^k z^j = \sum_{0 \leq j \leq k} \left\{ \begin{matrix} k \\ j \end{matrix} \right\}_{\alpha, \beta} z^j \times D_z^{(j)} \left[\frac{1 - z^{n+1}}{1 - z} \right].$$

5.3. **Some limitations on the convergence of the zeta series at $z = 1$.** Most of the generating functions we have employed in constructing the examples and applications within this article are based on variants of the geometric series, $G(z) = 1/(1 - cz)$, for some non-zero constant $c \in \mathbb{C}$ such that $|cz| < 1$ or when $cz \equiv -1$. One notable and obvious limitation of applying these geometric-series-based cases of our new transformations defined in Section 3 is that we are not able to handle series of the form

$\sum_n z^n / (\alpha n + \beta)^s$ when $|z| \equiv 1$, nor when $|\frac{1}{z}| < 1$. This restriction prevents us from constructing further examples of series for special zeta functions and constants such as

$$\frac{\pi^2}{8} = \sum_{k \geq 0} \frac{1}{(2k+1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

$$\zeta(s) = \frac{1}{1-2^{-s}} \times \sum_{n \geq 0} \frac{1}{(2n+1)^s}.$$

On the other hand, expansions for multiples of π^2 and *Catalan's constant*, G , for example, such as

$$\frac{\pi^2}{12} = \sum_{k \geq 0} \frac{(-1)^k}{(k+1)^2}$$

$$= \sum_{k \geq 0} (-1)^k \left[\frac{13}{(3k+1)^2} - \frac{13}{(3k+2)^2} + \frac{4}{(3k+3)^2} \right],$$

are readily handled by our new transformations of ordinary power series generating functions. The treatment given in Section 4.5.3 whereby we expand the ordinary, or classical, Hurwitz zeta function by its alternating series variant which we *are* readily able to handle with these new results suggests one workaround for the general geometric-series-based zeta function and polylogarithm cases.

Notes for readers and reviewers (supplementary computational data summary). A summary *Mathematica* notebook providing numerical data and supporting computations in deriving key results and new applications to specific series is prepared online at the following *Google Drive* link: <https://drive.google.com/file/d/0B6na6iIT7ICZMjJn0FcySmlBMGs/view?usp=sharing>. The intention of this supplementary document included with the submission of this article is to help the reviewer process the article more quickly, and to assist the reader with verifying and modifying the examples presented as applications of the new results cited above.

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