THE HOSOYA POLYNOMIAL, WIENER INDEX AND HYPER-WIENER INDEX OF JAHANGIR GRAPH $J_{8,m}$

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Abstract. Let $G(V,E)$ be simple connected graph with vertex set $V$ and edge set $E$. The Wiener index in the graph is $W(G) = \sum_{\{u,v\} \subseteq V} d(u,v)$, where $d(u,v)$ the distance between $u$ and $v$ and Hosoya polynomial of $G$ is $H(G,x) = \sum_{\{u,v\} \subseteq V} x^{d(u,v)}$, and hyper-Wiener index of $G$ is $WW(G) = \frac{1}{2}(W(G) + \sum_{\{u,v\} \subseteq V} d^2(u,v))$. In this paper we will compute the Wiener index, Hosoya polynomial and hyper-Wiener index of Jahangir graph $J_{8,m}$ for $m \geq 3$.

1. Introduction

Through this paper, we consider simple connected graph, i.e. connected without loops and multiple edges. Let $G(V,E)$ be a graph with vertex and edge are sets of $V(G)$ and $E(G)$, respectively. As usual, the distance between $u$ and $v$ of $G$ is denoted by $d(u,v)$, which is defined as the length of shortest path between $u$ and $v$ of $G$. The degree of vertex $u$ in $G$ is denoted by $d(u)$, which is defined as the number of edges incident to $u$. The diameter of $G$ is denoted by $D(G)$, which is defined as $D(G) = max_{u \in V}\{d(u,v) : v \in V\}$.

Jahangir graph $J_{n,m}$ is a graph on $nm + 1$ vertices and $m(n+1)$ edges for $n \geq 2$ and $m \geq 3$. $J_{n,m}$ consists of cycle $C_{nm}$ with one additional vertex referred by $O$ which is adjacent to $m$ vertices of $C_{nm}$ at distance to each other. $J_{n,m}$ consists of one vertex $O$ has degree $m$ and $n$ vertices have degree 3 and $m(n-1)$ vertices have degree 2.

The Wiener index of a graph is defined as $W(G) = \frac{1}{2} \sum_{u \in V} \sum_{v \in V} d(u,v)$ it was put forward in 1947 by the American physical chemist Harold Wiener. At first, Wiener index was used for predicting the boiling of paraffin [1], but later a strong correlation between the Wiener index and the chemical properties of a compound was found. Recent proposal to use in the prediction of conformational switching in RNA structures [2].

The Hosoya polynomial of graph is defined as $H(G,x) = \frac{1}{2} \sum_{u \in V} \sum_{v \in V} x^{d(u,v)}$ was introduced by Haruo Hosoya in 1988 as a counting polynomial it actually counts the number of distances of path of different lengths in a molecular graph [3]. Hosoya

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The hyper-Wiener index is defined as: 
$$W_G(G) = \frac{1}{2}(W(G) + \frac{1}{2}\sum_{u \in V} \sum_{v \in V} d^2(u,v)),$$
where $d^2(u,v) = (d(u,v))^2$.

The relationship between Wiener index and Hosoya polynomial is given as follows: 
$$W(G) = \frac{d}{dx}H(G,x)|_{x=1},$$
and the relationship between hyper-Wiener index and Hosoya polynomial is given as follows: 
$$W_G(G) = \frac{1}{2}\frac{d^2}{d^2x}H(G,x)|_{x=1}.$$

2. Notation and Terminology

Many researchers computed the Wiener index, Hosoya polynomial and hyper-Wiener index for $J_{n,m}$ with specific values for $n$. Now we will review the following results:

**Theorem 2.1** [7].
1. $H(J_{3,m}) = 4mx^1 + \frac{1}{2}m(m + 9)x^2 + 2m(m - 1)x^3 + m(2m - 5)x^4.
2. $W(J_{3,m}) = 15m^2 - 13m$.

**Theorem 2.2** [8].
1. $H(J_{4,m}) = 5mx^1 + \frac{1}{2}(m^2 + 11m)x^2 + (2m^2 + m)x^3 + (3m^2 - 4m)x^4 + (2m^2 - 4m)x^5 + \frac{1}{2}m(m - 3)x^6.
2. $W(J_{4,m}) = 32m^2 - 48m$.

**Theorem 2.3** [9].
1. $H(J_{5,m}) = 6mx^1 + \frac{1}{2}(m^2 + 13m)x^2 + (2m^2 + 5m)x^3 + (4m^2 - 4m)x^4 + (4m^2 - 6m)x^5 + (2m^2 - 3m)x^6.
2. $W(J_{5,m}) = 55m^2 - 42m$.

**Theorem 2.4** [10].
1. $H(J_{6,m}) = 7mx^1 + \left(\frac{m^2 + 13m}{2}\right)x^2 + (2m^2 + 6m)x^3 + 4m^2x^4 + (5m^2 - 6m)x^5 + (4m^2 - 6m)x^6 + (2m^2 - 4m)x^7 + \left(\frac{m^2 - 3m}{2}\right)x^8.
2. $W(J_{6,m}) = 90m^2 - 66m$.

**Theorem 2.5** [11].
1. $H(J_{7,m}) = 8mx^1 + \frac{1}{2}(m^2 + 17m)x^2 + (2m^2 + 7m)x^3 + (4m^2 + 5m)x^4 + (6m^2 - 6m)x^5 + (6m^2 - 8m)x^6 + (2m - 3)x^7 + (2m^2 - 5m)x^8.$
2- \( W(J_{7,m}) = 87m^2 - 14m - 21 \).
3- \( WW(J_{7,m}) = \frac{1}{2}(683m^2 - 513m - 168) \).

3. Main results

In this paper we will compute the Hosoya polynomial, Wiener index and hyper-Wiener index for \( J_{8,m} \) for an integer \( m \geq 3 \). From the definition of Jahangir graph we can see its graph consisting of cycle \( C_{8m} \) with an additional vertex referred by \( O \) adjacent to \( m \) vertices of \( C_{8m} \) at distance to each other. The order of vertex set \( V(J_{8,m}) \) is equal to \( 7m + m + 1 = 8m + 1 \), such that there are \( 7m \) vertices of Jahangir graph \( J_{8,m} \) have degree 2 and \( m \) vertices have degree 3 and one vertex \( O \) has degree \( m \). This imply that the size of edge set \( E(J_{8,m}) \) is \( \frac{2 \times 7m + 3 \times m + m \times 1}{2} = 9m \). To compute the Hosoya polynomial, Wiener index and hyper-Wiener index of \( J_{8,m} \) we first introduce some notes, which are useful to aims in this paper.

3.1. Notes.

1- We will divide \( J_{8,m} \) into \( m \) regions in each region there are 8 vertices and we will give them numbers from 1 to 8. In addition, we will give the regions numbers from 1 to \( m \) as shown in Figure 3.1.

![Figure 3.1: J_{8,m}](image)

2- The vertex \( v_a \) located in region \( b \) will be marked with the following symbol \( (v_a)_b \) for \( 1 \leq a \leq 8, 1 \leq b \leq m \) and the region \( b \) will be marked with the following symbol \( r_b \) for \( 1 \leq b \leq m \).
3- The number of unordered pairs of vertices \( x \) and \( y \) such that distances \( d(x, y) = k \) is denoted by \( d(J_{8, m}, k) \). Obviously, \( 1 \leq k \leq D(J_{8, m}) = 10 \), where \( D(J_{8, m}) = 10 \). For that, \( H(J_{8, m}, x) = \sum_{k=1}^{D(J_{8, m})} d(J_{8, m}, k)x^k = \sum_{k=1}^{10} d(J_{8, m}, k)x^k \).

4- We will divide the vertex set \( V(J_{8, m}) \) of \( J_{8, m} \) into several partitions based on \( d(v) \) and denoted by \( V_1, V_2 \) and \( V_m \), where \( V_k = \{ v \in V(J_{8, m}) : d(v) = k \} \), thus

\[
\begin{align*}
V_2 &= \{ v \in V(J_{8, m}) : d(v) = 2 \} \implies |V_2| = 7m. \\
V_3 &= \{ v \in V(J_{8, m}) : d(v) = 3 \} \implies |V_3| = m. \\
V_m &= \{ v \in V(J_{8, m}) : d(v) = m \} \implies |V_m| = 1.
\end{align*}
\]

5- We will rely on the tables for proofs in order to be brief, where the table shows begin and end of the pairs.

6- \( d(J_{8, m}, k) \) between \( v_i \) and any vertex of \( J_{8, m} \) for \( 2 \leq i \leq 4 \) is equal to \( d(J_{8, m}, k) \) between \( v_j \) and any vertices of \( J_{8, m} \) for \( 6 \leq j \leq 8 \), respectively. So, we will not but the vertices \( v_j \) for \( 6 \leq j \leq 8 \) in the tables that will appear in the proofs.

7- We will compute \( d(J_{8, m}, k) \) between \( (v_1)_i \) and all vertices of \( J_{8, m} \) for \( 1 \leq i \leq 5 \) then we will multiply the answer in \( m \) because we have \( m \) regions.

**Lemma 3.1.** \( d(J_{8, m}, 1) = E(J_{8, m}) = 9m. \)

Proof. It is obvious that the number of pairs from the length 1 is equal to the number of edges.

**Lemma 3.2.** \( d(J_{8, m}, 2) = \frac{1}{2}(m^2 + 19m). \)

Proof.

<table>
<thead>
<tr>
<th>( d(J_{8, m}, 2) )</th>
<th>The begin</th>
<th>The end</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2m )</td>
<td>( O \in V_m )</td>
<td>( v_2, v_8 ) in each ( r_i : 1 \leq i \leq m )</td>
</tr>
<tr>
<td>( 2 )</td>
<td>( (v_1)_1 \in V_3 )</td>
<td>( (v_3)_1, (v_7)_m )</td>
</tr>
<tr>
<td>( m - 1 )</td>
<td>( (v_1)_1 \in V_3 )</td>
<td>( v_1 ) in each ( r_i : 2 \leq i \leq m )</td>
</tr>
<tr>
<td>( 3 )</td>
<td>( (v_2)_1 \in V_2 )</td>
<td>( (v_4)_1, (v_8)_m, O )</td>
</tr>
<tr>
<td>( 2 )</td>
<td>( (v_3)_1 \in V_2 )</td>
<td>( (v_5)_1, (v_1)_1 )</td>
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<tr>
<td>( 2 )</td>
<td>( (v_4)_1 \in V_2 )</td>
<td>( (v_6)_1, (v_2)_1 )</td>
</tr>
<tr>
<td>( 2 )</td>
<td>( (v_5)_1 \in V_2 )</td>
<td>( (v_7)_1, (v_3)_1 )</td>
</tr>
</tbody>
</table>

\[
d(J_{8, m}, 2) = \frac{1}{2}(1(2m) + m(2 + m - 1) + m(3) + m(2) + m(2) + m(2) + m(2) + m(2) + m(3)) \]
\[
= \frac{1}{2}(m^2 + 19m).
\]

**Lemma 3.3.** \( d(J_{8, m}, 3) = 2m^2 + 8m. \)

Proof.

}\]
Lemma 3.4. $d(J_{8,m}, 4) = 4m^2 + 6m$.
Proof.

\[
d(J_{8,m}, 4) = \frac{1}{2}(1(2m) + m(4 + 2(m - 2)) + m(2 + m - 1) + m(3) + m(2) + m(2 + m) + m(3) + m(2 + m - 1))
\]

\[
= 4m^2 + 6m.
\]
\[
d(J_{8,m}, 5) = \frac{1}{2} (1(m) + m(2 + 2(m - 2)) + m(3 + 2(m - 2)) + m(3 + 2(m - 2))) + m(1 + m - 1) + m(3) + m(1 + m - 1) + m(3 + 2(m - 2)) + m(3 + 2(m - 2))) = 6m^2 - m.
\]

**Lemma 3.6.** \(d(J_{8,m}, 6) = 7m^2 - 8m.\)

**Proof.**

\[
d(J_{8,m}, 6) = \frac{1}{2} (m(m - 2) + m(1 + 2(m - 2)) + m(2 + 2(m - 2)) + m(2 + 2(m - 2))) + m(2 + m - 2) + m(2 + 2(m - 2)) + m(2 + 2(m - 2)) + m(1 + 2(m - 2))) = 7m^2 - 8m.
\]

**Lemma 3.7.** \(d(J_{8,m}, 7) = 6m^2 - 8m.\)

**Proof.**
\[d(J_{8,m}, 7) = \frac{1}{4}(m(m-2) + m(1 + 2(m-2)) + m(4 + 2(m-3)) + m(2 + 2(m-2)) + m(1 + 2(m-2)) + m(m-2)) \]
\[= 6m^2 - 8m.\]

**Lemma 3.8.** \(d(J_{8,m}, 8) = 4m^2 - 6m.\)

**Proof.**

\[
d(J_{8,m}, 8) = \frac{1}{4}(m(m-2) + m(1 + 2(m-2)) + m(4 + 2(m-3)) + m(1 + 2(m-2)) + m(m-2)) \]
\[= 4m^2 - 6m.\]

**Lemma 3.9.** \(d(J_{8,m}, 9) = 2m^2 - 4m.\)

**Proof.**

\[
d(J_{8,m}, 9) = \frac{1}{4}(m(m-2) + m(2 + 2(m-3)) + m((m-2)) \]
\[= 2m^2 - 4m.\]

**Lemma 3.10.** \(d(J_{8,m}, 10) = \frac{1}{2}(m^2 - 3m).\)

**Proof.**

\[
d(J_{8,m}, 10) = \frac{1}{2}(m(m-3)) \]
\[= \frac{1}{2}(m^2 - 3m).\]
Theorem 3.1.
\[ H(J_{8,m}, x) = 9mx^1 + \frac{1}{2}(m^2 + 19m)x^2 + (2m^2 + 8m)x^3 + (4m^2 + 6m)x^4 + (6m^2 - m)x^5 + (7m^2 - 8m)x^6 + (6m^2 - 8m)x^7 + (4m^2 - 6m)x^8 + (2m^2 - 4m)x^9 + \frac{1}{2}(m^2 - 3m)x^{10}. \]
Proof.
\[ H(J_{8,m}, x) = \sum_{k=1}^{D(J_{8,m})} d(J_{8,m}, k) x^k = \sum_{k=1}^{10} d(J_{8,m}, k) x^k = 9mx^1 + \frac{1}{2}(m^2 + 19m)x^2 + (2m^2 + 8m)x^3 + (4m^2 + 6m)x^4 + (6m^2 - m)x^5 + (7m^2 - 8m)x^6 + (6m^2 - 8m)x^7 + (4m^2 - 6m)x^8 + (2m^2 - 4m)x^9 + \frac{1}{2}(m^2 - 3m)x^{10}. \]

Theorem 3.2.
\[ W(J_{8,m}) = 192m^2 - 132m. \]
Proof.
\[ W(J_{8,m}) = \frac{d}{dx} H(J_{8,m}, x) \bigg|_{x=1} = \sum_{k=1}^{10} d(J_{8,m}, k) k x^k = 9m + \frac{1}{2}(m^2 + 19m) + 3(2m^2 + 8m) + 4(4m^2 + 6m) + 5(6m^2 - m) + 6(7m^2 - 8m) + 7(6m^2 - 8m) + 8(4m^2 - 6m) + 9(2m^2 - 4m) + \frac{10}{2}(m^2 - 3m) = 192m^2 - 132m. \]

Theorem 3.3.
\[ WW(J_{8,m}) = 720m^2 - 740m. \]
Proof.
\[ WW(J_{8,m}) = \frac{1}{2} \frac{d^2}{dx^2} x H(J_{8,m}, x) \bigg|_{x=1} = \frac{1}{2} \frac{d^2}{dx^2} [9mx^2 + \frac{1}{2}(m^2 + 19m)x^3 + (2m^2 + 8m)x^4 + (4m^2 + 6m)x^5 + (6m^2 - m)x^6 + (7m^2 - 8m)x^7 + (6m^2 - 8m)x^8 + (4m^2 - 6m)x^9 + (2m^2 - 4m)x^{10} + \frac{1}{2}(m^2 - 3m)x^{11}] \bigg|_{x=1} = \frac{1}{2} [9(2)(1)m + \frac{1}{2}(3)(2)(m^2 + 19m)x^1 + (4)(3)(2m^2 + 8m)x^2 + (5)(4)(4m^2 + 6m)x^3 + (6)(5)(6m^2 - m)x^4 + (7)(6)(7m^2 - 8m)x^5 + (8)(7)(6m^2 - 8m)x^6 + (9)(8)(4m^2 - 6m)x^7 + (10)(9)(2m^2 - 4m)x^8 + \frac{1}{2}(11)(10)(m^2 - 3m)x^9 = 720m^2 - 740m. \]

4. Conclusion

In this paper, we computed Hosoya polynomial of \( J_{8,m} \) for any value of \( m \geq 3 \). In addition, depending on Hosoya polynomial we computed Wiener index and hyper-Wiener index of \( J_{8,m} \) for any value of \( m \geq 3 \).
References


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