THE HOSOYA POLYNOMIAL, WIENER INDEX AND HYPER-WIENER INDEX OF JAHANGIR GRAPH $J_{8,m}$

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Abstract. Let $G(V,E)$ be simple connected graph with vertex set V and edge set E. The Wiener index in the graph is $W(G) = \sum_{\{u,v\}\subseteq V} d(u,v)$, where $d(u,v)$ the distance between u and v and Hosoya polynomial of G is $H(G,x) = \sum_{\{u,v\}\subseteq V} x^{d(u,v)}$, and hyper-Wiener index of G is $WW(G) = \frac{1}{2}(W(G) + \sum_{\{u,v\}\subseteq V} d^2(u,v)$. In this paper we will compute the Wiener index, Hosoya polynomial and hyper-Wiener index of Jahangir graph $J_{8,m}$ for $m \geq 3$.

1. Introduction

Through this paper, we consider simple connected graph, i.e. connected without loops and multiple edges. Let $G(V,E)$ be a graph with vertex and edge are sets of $V(G)$ and $E(G)$, respectively. As usual, the distance between u and v of G is denoted by $d(u,v)$, which is defined as the length of shortest path between u and v of G. The degree of vertex u in G is denoted by $d(u)$, which is defined as the number of edges incident to u. The diameter of G is denoted by $D(G)$, which is defined as $D(G) = max_{u\epsilon V}\{d(u,v) : v\epsilon V\}$.

Jahangir graph $J_{n,m}$ is a graph on $nm+1$ vertices and $m(n+1)$ edges for $n \geq 2$ and $m \geq 3$. $J_{n,m}$ consists of cycle C_{nm} with one additional vertex referred by O which is adjacent to m vertices of C_{nm} at distance to each other. $J_{n,m}$ consists of one vertex O has degree m and n vertices have degree 3 and $m(n-1)$ vertices have degree 2.

The Wiener index of a graph is defined as $W(G) = \frac{1}{2}\sum_{u\epsilon V}\sum_{v\epsilon V} d(u,v)$ it was put forward in 1947 by the American physical chemist Harold Wiener. At first, Wiener index was used for predicting the boiling of paraffin [1], but later a strong correlation between the Wiener index and the chemical properties of a compound was found. Recent proposal to use in the prediction of conformational switching in RNA structures [2].

The Hosoya polynomial of graph is defined as $H(G,x) = \frac{1}{2}\sum_{u\epsilon V}\sum_{v\epsilon V} x^{d(u,v)}$ was introduced by Haruo Hosoya in 1988 as a counting polynomial it actually counts the number of distances of path of different lengths in a molecular graph [3]. Hosoya

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polynomial is very well studied. In 1993, Gutman introduced Hosoya polynomial for a vertex of a graph [4]; and these polynomials are correlated. The most interesting application of the Hosoya polynomial is the almost all distance based graph invariants, which are used to predict physical, chemical and pharmacological properties of organic molecules, can be recovered from the Hosoya polynomial [5].

The hyper-Wiener index of acyclic graph was introduced by Randić 1993 then, Klein et al. [6] generalized Randi definition for all connected graph, as a generalization of the Wiener index. The hyper-Wiener index is defined as: $WW(G) = \frac{1}{2}(W(G) + \frac{1}{2}\sum_{u\epsilon V}\sum_{v\epsilon V} d^2(u,v))$, where $d^2(u,v) = (d(u,v))^2$.

The relationship between Wiener index and Hosoya polynomial is given as follows: $W(G) = \frac{\partial}{\partial x}H(G,x) \mid_{x=1}$, The relationship between hyper-Wiener index and Hosoya polynomial is given as follows: $WW(G) = \frac{1}{2}\frac{\partial^2}{\partial x^2}H(G,x) \mid_{x=1}$.

2. Notation and Terminology

Many researchers computed the Wiener index, Hosoya polynomial and hyper-Wiener index for $J_{n,m}$with specific values for n. Now we will review the following results:

Theorem 2.1 [7].
1- $H(J_{3,m},x) = 4mx^1 + \frac{1}{2}m(m+9)x^2 + 2m(m-1)x^3 + m(2m-5)x^4$.
2- $W(J_{3,m}) = 15m^2 - 13m$.

Theorem 2.2 [8].
1- $H(J_{4,m},x) = 5mx^1 + \frac{1}{2}(m^2+11m)x^2 + (2m^2+m)x^3 + (3m^2-4m)x^4$
$+(2m^2-4m)x^5 + \frac{1}{2}m(m-3)x^6$
2- $W(\ J_{4,m}) = 32m^2 - 48m$.

Theorem 2.3 [9].
1- $H(J_{5,m},x) = 6mx^1 + \frac{1}{2}(m^2+13m)x^2 + (2m^2+5m)x^3 + (4m^2-4m)x^4$
$+(4m^2-6m)x^5 + (2m^2-3m)x^6$.
2- $W(J_{5,m}) = 55m^2 - 42m$.

Theorem 2.4 [10].
1- $H(J_{6,m},x) = 7mx^1 + (\frac{m^2+13m}{2})x^2 + (2m^2+6m)x^3 + 4m^2x^4 + (5m^2-6m)x^5$
$+(4m^2-6m)x^6 + (2m^2-4m)x^7 + (\frac{m^2-3m}{2})x^8$.
2- $W(J_{6,m}) = 90m^2 - 66m$.

Theorem 2.5 [11].
1- $H(J_{7,m},x) = 8mx^1 + \frac{1}{2}(m^2+17m)x^2 + (2m^2+7m)x^3 + (4m^2+5m)x^4$
$+(6m^2-6m)x^5 + (6m^2-8m)x^6 + (2m-3)x^7 + (2m^2-5m)x^8$.

2- $W(J_{7,m}) = 87m^2 - 14m - 21$.
3- $WW(J_{7,m}) = \frac{1}{2}(683m^2 - 513m - 168)$.

3. Main results

In this paper we will compute the Hosoya polynomial, Wiener index and hyper-Wiener index for $J_{8,m}$ for an integer $m \geq 3$. From the definition of Jahangir graph we can see its graph consisting of cycle C_{8m} with on additional vertex referred by O adjacent to m vertices of C_{8m} at distance to each other. The order of vertex set $V(J_{8,m})$ is equal to $7m + m + 1 = 8m + 1$, such that there are $7m$ vertices of Jahangir graph $J_{8,m}$ have degree 2 and m vertices have degree 3 and one vertex O has degree m. This imply that the size of edge set $E(J_{8,m})$ is $\frac{2\times 7m+3\times m+m\times 1}{2} = 9m$. To compute the Hosoya polynomial, Wiener index and hyper-Wiener index of $J_{8,m}$ we first introduce some notes, which are useful to aims in this paper.

3.1. **Notes.**

1- We will divide $J_{8,m}$ into m regions in each region there are 8 vertices and we will give them numbers from 1 to 8. In addition, we will give the regions numbers from 1 to m as shown in Figure 3.1.

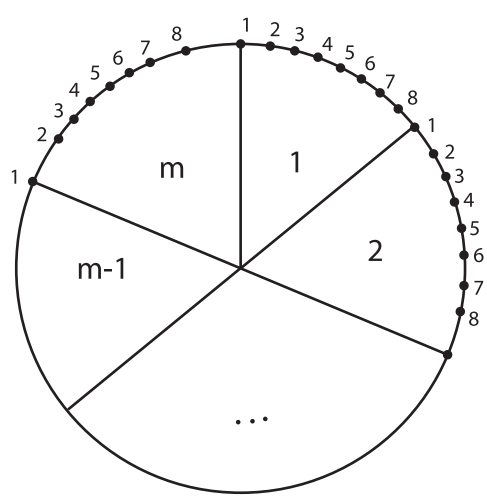

Figure 3.1: $J_{8,m}$

2- The vertex v_a located in region b will be marked with the following symbol $(v_a)_b$ for $1 \leq a \leq 8, 1 \leq b \leq m$ and the region b will be marked with the following symbol r_b for $1 \leq b \leq m$.

3- The number of unordered pairs of vertices x and y such that distances $d(x,y)=k$ is denoted by $d(J_{8,m},k)$. Obviously, $1 \leq k \leq D(J_{8,m})=10$, where $D(J_{8,m})=10$. for that, $H(J_{8,m},x)=\sum_{k=1}^{D(J_{8,m})} d(J_{8,m},k)x^k=\sum_{k=1}^{10} d(J_{8,m},k)x^k$.

4- We will divide the vertex set $V(J_{8,m})$ of $J_{8,m}$ into several partitions based on $d(v)$ and denoted by V_1,V_2 and V_m, where $V_k=\{v \in V(J_{8,m}) : d(v)=k\}$, thus
$V_2=\{v \in V(J_{8,m}) : d(v)=2\} \longrightarrow \mid V_2 \mid = 7m$.
$V_3=\{v \in V(J_{8,m}) : d(v)=3\} \longrightarrow \mid V_3 \mid = m$.
$V_m=\{v \in V(J_{8,m}) : d(v)=m\} \longrightarrow \mid V_m \mid = 1$.

5- We will rely on the tables for proofs in order to be brief, where the table shows begin and end of the pairs.

6- $d(J_{8,m},k)$ between v_i and any vertex of $J_{8,m}$ for $2 \leq i \leq 4$ is equal to $d(J_{8,m},k)$ between v_j and any vertices of $J_{8,m}$ for $6 \leq j \leq 8$, respectively. So, we will not but the vertices v_j for $6 \leq j \leq 8$ in the tables that will appear in the proofs.

7- We will compute $d(J_{8,m},k)$ between $(v_i)_1$ and all vertices of $J_{8,m}$ for $1 \leq i \leq 5$ then we will multiply the answer in m because we have m regions.

Lemma 3.1. $d(J_{8,m},1)=E(J_{8,m})=9m$.
Proof. It is obvious that the number of pairs from the length 1 is equal to the number of edges.

Lemma 3.2. $d(J_{8,m},2)=\frac{1}{2}(m^2+19m)$.
Proof.

$d(J_{8,m},2)$	The begin	The end
$2m$	$O \in V_m$	v_2,v_8 in each $r_i : 1 \leq i \leq m$
2	$(v_1)_1 \in V_3$	$(v_3)_1,(v_7)_m$
$m-1$	$(v_1)_1 \in V_3$	$v_1$in each $r_i : 2 \leq i \leq m$
3	$(v_2)_1 \in V_2$	$(v_4)_1,(v_8)_m,O$
2	$(v_3)_1 \in V_2$	$(v_5)_1,(v_1)_1$
2	$(v_4)_1 \in V_2$	$(v_6)_1,(v_2)_1$
2	$(v_5)_1 \in V_2$	$(v_7)_1,(v_3)_1$

$$d(J_{8,m},2)=\frac{1}{2}(1(2m)+m(2+m-1)+m(3)+m(2)+m(2)$$
$$+m(2)+m(2)+m(2)+m(3))$$
$$=\frac{1}{2}(m^2+19m).$$

Lemma 3.3. $d(J_{8,m},3)=2m^2+8m$.
Proof.

$d(J_{8,m},3)$	The begin	The end
$2m$	$O \in V_m$	v_3, v_7 in each $r_i : 1 \leq i \leq m$
4	$(v_1)_1 \in V_3$	$(v_4)_1, (v_8)_1, (v_6)_m, (v_2)_m$
$2(m-2)$	$(v_1)_1 \in V_3$	v_2, v_8 in each $r_i : 2 \leq i \leq m-1$
2	$(v_2)_1 \in V_2$	$(v_5)_1, (v_7)_m$
$m-1$	$(v_2)_1 \in V_2$	$v_1$in each $r_i : 2 \leq i \leq m$
3	$(v_3)_1 \in V_2$	$(v_6)_1, (v_8)_m, O$
2	$(v_4)_1 \in V_2$	$(v_7)_1, (v_1)_1$
2	$(v_5)_1 \in V_2$	$(v_8)_1, (v_2)_1$

$$d(J_{8,m},3) = \tfrac{1}{2}(1(2m) + m(4+2(m-2)) + m(2+m-1) + m(3)$$
$$+m(2) + m(2) + m(2) + m(3) + m(2+m-1))$$
$$= 2m^2 + 8m.$$

Lemma 3.4. $d(J_{8,m},4) = 4m^2 + 6m$.
Proof.

$d(J_{8,m},4)$	The begin	The end
$2m$	$O \in V_m$	v_4, v_6 in each $r_i : 1 \leq i \leq m$
4	$(v_1)_1 \in V_3$	$(v_5)_1, (v_7)_1, (v_3)_m, (v_5)_m$
$2(m-2)$	$(v_1)_1 \in V_3$	v_3, v_8 in each $r_i : 2 \leq i \leq m-1$
4	$(v_2)_1 \in V_2$	$(v_6)_1, (v_8)_1, (v_6)_m, (v_2)_m$
$2(m-2)$	$(v_2)_1 \in V_2$	v_2, v_8 in each $r_i : 2 \leq i \leq m-1$
2	$(v_3)_1 \in V_2$	$(v_7)_1, (v_7)_m$
$m-1$	$(v_3)_1 \in V_2$	v_1 in each $r_i : 2 \leq i \leq m$
3	$(v_4)_1 \in V_2$	$(v_8)_1, (v_8)_m, O$
2	$(v_5)_1 \in V_2$	$(v_1)_1, (v_1)_2$

$$d(J_{8,m},4) = \tfrac{1}{2}(1(2m) + m(4+2(m-2)) + m(4+2(m-2)) + m(2+m-1)$$
$$+m(3) + m(2) + m(3) + m(2+m-1) + m(4+2(m-2)))$$
$$= 4m^2 + 6m.$$

Lemma 3.5. $d(J_{8,m},5) = 6m^2 - m$.
Proof.

$d(J_{8,m},5)$	The begin	The end
m	$O \in V_m$	v_5 in each $r_i : 1 \leq i \leq m$
2	$(v_1)_1 \in V_3$	$(v_6)_1, (v_4)_m$
$2(m-2)$	$(v_1)_1 \in V_3$	v_4, v_6 in each $r_i : 2 \leq i \leq m-1$
3	$(v_2)_1 \in V_2$	$(v_7)_1, (v_5)_m, (v_3)_m$
$2(m-2)$	$(v_2)_1 \in V_2$	v_3, v_7 in each $r_i : 2 \leq i \leq m-1$
3	$(v_3)_1 \in V_2$	$(v_8)_1, (v_6)_m, (v_2)_m$
$2(m-2)$	$(v_3)_1 \in V_2$	v_2, v_8 in each $r_i : 2 \leq i \leq m-1$
1	$(v_4)_1 \in V_2$	$(v_7)_m$
$m-1$	$(v_4)_1 \in V_2$	v_1 in each $r_i : 2 \leq i \leq m$
3	$(5_4)_1 \in V_2$	$(v_2)_2, (v_8)_m, O$

$$d(J_{8,m},5) = \frac{1}{2}(1(m) + m(2+2(m-2)) + m(3+2(m-2)) + m(3+2(m-2)) \\ + m(1+m-1) + m(3) + m(1+m-1) + m(3+2(m-2)) + m(3+2(m-2))) \\ = 6m^2 - m.$$

Lemma 3.6. $d(J_{8,m},6) = 7m^2 - 8m$.

Proof.

$d(J_{8,m},6)$	The begin	The end
$m-2$	$(v_1)_1 \in V_3$	v_5 in each $r_i : 2 \leq i \leq m-1$
1	$(v_2)_1 \in V_2$	$(v_4)_m$
$2(m-2)$	$(v_2)_1 \in V_2$	v_4, v_6 in each $r_i : 2 \leq i \leq m-1$
2	$(v_3)_1 \in V_2$	$(v_3)_m, (v_5)_m$
$2(m-2)$	$(v_3)_1 \in V_2$	v_3, v_7 in each $r_i : 2 \leq i \leq m-1$
2	$(v_4)_1 \in V_2$	$(v_6)_m, (v_2)_m$
$2(m-2)$	$(v_4)_1 \in V_2$	v_2, v_8 in each $r_i : 2 \leq i \leq m-1$
2	$(v_5)_1 \in V_2$	$(v_3)_2, (v_7)_m$
$m-2$	$(v_5)_1 \in V_2$	v_1 in each $r_i : 3 \leq i \leq m$

$$d(J_{8,m},6) = \frac{1}{2}(m(m-2) + m(1+2(m-2)) + m(2+2(m-2)) + m(2+2(m-2)) \\ + m(2+m-2) + m(2+2(m-2)) + m(2+2(m-2)) + m(1+2(m-2))) \\ = 7m^2 - 8m.$$

Lemma 3.7. $d(J_{8,m},7) = 6m^2 - 8m$.
Proof.

$d(J_{8,m},7)$	The begin	The end
$m-2$	$(v_2)_1 \in V_2$	v_5 in each $r_i : 2 \leq i \leq m-1$
1	$(v_3)_1 \in V_2$	$(v_4)_m$
$2(m-2)$	$(v_3)_1 \in V_2$	v_4, v_6in each $r_i : 2 \leq i \leq m-1$
2	$(v_4)_1 \in V_2$	$(v_5)_m, (v_3)_m$
$2(m-2)$	$(v_4)_1 \in V_2$	v_3, v_7 in each $r_i : 2 \leq i \leq m-1$
4	$(v_5)_1 \in V_2$	$(v_6)_m, (v_2)_m, (v_4)_2, (v_8)_2$
$2(m-3)$	$(v_5)_1 \in V_2$	v_2, v_8 in each $r_i : 3 \leq i \leq m-1$

$d(J_{8,m},7) = \frac{1}{2}(m(m-2) + m(1+2(m-2)) + m(2+2(m-2)) + m(4+2(m-3))$
$+m(2+2(m-2)) + m(1+2(m-2)) + m(m-2))$
$= 6m^2 - 8m.$

Lemma 3.8. $d(J_{8,m},8) = 4m^2 - 6m.$
Proof.

$d(J_{8,m},8)$	The begin	The end
$m-2$	$(v_3)_1 \in V_2$	v_5 in each $r_i : 2 \leq i \leq m-1$
1	$(v_4)_1 \in V_2$	$(v_4)_m$
$2(m-2)$	$(v_4)_1 \in V_2$	v_4, v_6 in each $r_i : 2 \leq i \leq m-1$
4	$(v_5)_1 \in V_2$	$(v_5)_m, (v_3)_m, (v_5)_2, (v_7)_2$
$2(m-3)$	$(v_5)_1 \in V_2$	v_3, v_7 in each $r_i : 3 \leq i \leq m-1$

$d(J_{8,m},8) = \frac{1}{2}(m(m-2) + m(1+2(m-2)) + m(4+2(m-3))$
$+m(1+2(m-2)) + m(m-2))$
$= 4m^2 - 6m.$

Lemma 3.9. $d(J_{8,m},9) = 2m^2 - 4m.$
Proof.

$d(J_{8,m},9)$	The begin	The end
$m-2$	$(v_4)_1 \in V_2$	v_5 in each $r_i : 2 \leq i \leq m-2$
1	$(v_5)_1 \in V_2$	$(v_4)_m, (v_6)_2$
$2(m-3)$	$(v_5)_1 \in V_2$	v_4, v_6 in each $r_i : 3 \leq i \leq m-1$

$d(J_{8,m},9) = \frac{1}{2}(m(m-2) + m(2+2(m-3)) + m((m-2))$
$= 2m^2 - 4m.$

Lemma 3.10. $d(J_{8,m},10) = \frac{1}{2}(m^2 - 3m).$
Proof.

$d(J_{8,m},10)$	The begin	The end
$m-3$	$(v_5)_1 \in V_2$	v_5 in each $r_i : 3 \leq i \leq m-1$

$d(J_{8,m},10) = \frac{1}{2}(m(m-3))$
$= \frac{1}{2}(m^2 - 3m).$

Theorem 3.1.
$H(J_{8,m},x) = 9mx^1 + \frac{1}{2}(m^2+19m)x^2 + (2m^2+8m)x^3 + (4m^2+6m)x^4$
$+(6m^2-m)x^5 + (7m^2-8m)x^6 + (6m^2-8m)x^7 + (4m^2-6m)x^8$
$+(2m^2-4m)x^9 + \frac{1}{2}(m^2-3m)x^{10}$.
Proof.
$H(J_{8,m},x) = \sum_{k=1}^{D(J_{8,m})} d(J_{8,m},k)x^k$
$= \sum_{k=1}^{10} d(J_{8,m},k)x^k$
$= 9mx^1 + \frac{1}{2}(m^2+19m)x^2 + (2m^2+8m)x^3 + (4m^2+6m)x^4$
$+(6m^2-m)x^5 + (7m^2-8m)x^6 + (6m^2-8m)x^7 + (4m^2-6m)x^8$
$+(2m^2-4m)x^9 + \frac{1}{2}(m^2-3m)x^{10}$.

Theorem 3.2.
$W(J_{8,m}) = 192m^2 - 132m$.
Proof.
$W(J_{8,m}) = \frac{\partial}{\partial x}H(J_{8,m},x) \mid_{x=1}$
$= \sum_{k=1}^{10} d(J_{8,m},k) \times k$
$= 9m + \frac{2}{2}(m^2+19m) + 3(2m^2+8m) + 4(4m^2+6m) + 5(6m^2-m)$
$+6(7m^2-8m) + 7(6m^2-8m) + 8(4m^2-6m) + 9(2m^2-4m) + \frac{10}{2}(m^2-3m)$
$= 192m^2 - 132m$.

Theorem 3.3.
$WW(J_{8,m}) = 720m^2 - 740m$.
Proof.
$WW(J_{8,m}) = \frac{1}{2}\frac{\partial^2}{\partial x^2}xH(J_{8,m},x) \mid_{x=1}$
$= \frac{1}{2}\frac{\partial^2}{\partial x^2}[9mx^2 + \frac{1}{2}(m^2+19m)x^3 + (2m^2+8m)x^4 + (4m^2+6m)x^5$
$+(6m^2-m)x^6 + (7m^2-8m)x^7 + (6m^2-8m)x^8 + (4m^2-6m)x^9$
$+(2m^2-4m)x^{10} + \frac{1}{2}(m^2-3m)x^{11}] \mid_{x=1}$
$= \frac{1}{2}[9(2)(1)m + \frac{1}{2}(3)(2)(m^2+19m)x^1 + (4)(3)(2m^2+8m)x^2$
$+(5)(4)(4m^2+6m)x^3 + (6)(5)(6m^2-m)x^4 + (7)(6)(7m^2-8m)x^5$
$+(8)(7)(6m^2-8m)x^6 + (9)(8)(4m^2-6m)x^7 + (10)(9)(2m^2-4m)x^8$
$+\frac{1}{2}(11)(10)(m^2-3m)x^9$
$= 720m^2 - 740m$.

4. Conclusion

In this paper, we computed Hosoya polynomial of $J_{8.m}$ for any value of $m \geq 3$. In addition, depending on Hosoya polynomial we computed Wiener index and hyper-Wiener index of $J_{8,m}$ for any value of $m \geq 3$.

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