CONVOLUTION CONDITIONS FOR THE q-ANALOGUE CLASSES OF JANOWSKI FUNCTIONS

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ABSTRACT. Convolution conditions are discussed for the q-analogue classes of Janowski starlike, convex and spirallike functions.

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1. INTRODUCTION

Ruscheweyh and Shell-Small [7] while proving the Pólya-Schoenberg conjecture [6], used an ingenious and intricate argument which generated a wealth of results in convolutions. One such result is that of Ganesan [3] where convolution conditions for certain subclasses of analytic functions related to the Janowski classes defined using subordination, a concept which can be traced to Lindelöf [5]. In this note we give characterizations for q-analogue classes related to the Janowski class in terms of convolution. The intrinsic properties of q-analogues including the applications in the study of quantum groups and q-deformed subalgebras and of fractals are known in the literature. Some integral transforms in classical analysis have their q-analogues in the theory of q-calculus. This has led various researchers in the field of q-theory to extending important results in classical analysis to their q-analogues.
Let $\mathcal{A}$ denote the class of functions of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\},$$

Definition 1.1. *Let $q \in (0,1)$ and let $n \in \mathbb{C}$. The q-number, denoted by $[n]_q$, is defined as $[n]_q = \frac{1-q^n}{1-q}$. In the case when $n \in \mathbb{N}$, we obtain $[n]_q = 1 + q + q^2 + q^3 + ... + q^{n-1}$ and when $q \to 1^-$, $[n]_q = n$.*

Definition 1.2. [1] *The Jackson q-derivative of a function $f \in \mathcal{A}$ is defined by*

$$D_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{z(1-q)}, & z \neq 0, \\ f'(0), & z = 0. \end{cases} \quad , \quad \text{where } (0 < q < 1) \tag{1.2}$$

and $D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \quad z \neq 0.$

As a right inverse, Jackson [2] presented the q-integral of a function f as

$$\int_0^z f(t) d_q t = z(1-q) \sum_{n=0}^{\infty} q^n f(zq^n),$$

provided that the series converges. For a function $f(z) = z^n$, we note that

$$\int_0^z f(t) d_q t = \int_0^z t^n d_q t = \frac{z^{n+1}}{[n+1]_q} \qquad (n \neq -1)$$

and

$$\lim_{q \to 1^-} \int_0^z f(t) d_q t = \lim_{q \to 1^-} \frac{z^{n+1}}{[n+1]_q} = \frac{z^{n+1}}{n+1} = \int_0^z f(t) dt,$$

where $\int_0^z f(t)dt$ is the ordinary integral.

Under the hypothesis of the definition of q-difference operator, we have the following rules.

(i) $D_q(af(z) \pm bg(z)) = aD_q f(z) \pm bD_q g(z)$, where a and b any real (or complex) constants

(ii) $D_q(f(z)g(z)) = g(qz)D_q f(z) + f(z)D_q g(z) = f(z)D_q g(z) + D_q f(z)g(qz)$

(iii) $D_q\left(\frac{f(z)}{g(z)}\right) = \frac{g(z)D_q f(z) - f(z)D_q g(z)}{g(qz)g(z)}$.

Definition 1.3. *For two functions f and g analytic in $\mathcal{U}$, we say that the function f is subordinate to g in $\mathcal{U}$ and write $f(z) \prec g(z)$, if there exists a Schwarz function ω, which is analytic in $\mathcal{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that $f(z) = g(\omega(z))$, $z \in \mathcal{U}$.*

Definition 1.4. *For real numbers A, B, $-1 \leq B < A \leq 1$, $p \in P(A, B)$ if and only if*

$$p(z) \prec \frac{1 + Az}{1 + Bz}, \; z \in \mathcal{U}.$$

Definition 1.5. *A function $f(z) \in \mathcal{A}$ is said to be in the class $C_q(A, B)$ if and only if*

$$\frac{D_q(zD_q f(z))}{D_q f(z)} \in P(A, B).$$

Definition 1.6. *A function $f(z) \in \mathcal{A}$ is said to be in the class $S_q^*(A, B)$ if and only if*

$$\frac{zD_q f(z)}{f(z)} \in P(A, B).$$

Definition 1.7. *The convolution or Hadamard product, of two analytic functions*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \; (|z| < R_1) \text{ and } g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \; (|z| < R_2),$$

is defined as the power series

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \qquad |z| < R_1 R_2.$$

It can be easily seen that

$$zD_q f * g = f * zD_q g. \tag{1.3}$$

2. Main results

Theorem 2.1. *The function $f \in C_q(A, B)$ in $|z| < R \leq 1$ if and only if*

$$\frac{1}{z}\left[f * \frac{xz + \left(x + \frac{[2]_q(1+Ax)}{B-A}\right) qz^2 + \frac{(1+q-[2]_q)(1+Ax)}{B-A} qz^3}{(1-z)(1-qz)(1-q^2 z)} \right] \neq 0$$

Proof. The function $f \in C_q(A,B)$ if and only if

$$\frac{D_q(zD_qf(z))}{D_qf(z)} \in P(A,B), \qquad for\ all\ z \in \mathcal{U}. \tag{2.1}$$

Since $\frac{D_q(zD_qf)}{D_qf} = 1$ at $z = 0$, so (2.1) is equivalent to

$$\frac{D_q\left(zD_qf\right)}{D_qf} \neq \frac{1+Ax}{1+Bx}, \ (|z| < R, \ |x| = 1, \ x \neq -1)$$

which implies

$$(1+Bx)D_q\left(zD_qf\right) - (1+Ax)D_qf \neq 0. \tag{2.2}$$

Setting $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, we have

$$D_qf = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}$$

$$D_q\left(zD_qf\right) = 1 + \sum_{n=2}^{\infty} [n]_q^2 a_n z^{n-1} = D_qf * \frac{1}{(1-z)(1-qz)}.$$

The left hand side of (2.2) is equivalent to

$$\begin{aligned}
(1+Bx)\left[D_qf * \sum_{n=1}^{\infty}[n]_q z^{n-1}\right] &- D_qf * \sum_{n=1}^{\infty}(1+Ax)z^{n-1} \\
&= D_qf * \sum_{n=1}^{\infty}\left[(1+Bx)[n]_q - (1+Ax)\right] z^{n-1} \\
&= D_qf * \left(\frac{-(1+Ax)}{1-z} + \frac{1+Bx}{(1-z)(1-qz)}\right) \\
&= D_qf * \left(\frac{x(B-A) + (1+Ax)qz}{(1-z)(1-qz)}\right).
\end{aligned}$$

Thus

$$\frac{1}{z}\left[zD_qf * \frac{xz + \frac{(1+Ax)}{B-A}qz^2}{(1-z)(1-qz)}\right] \neq 0. \tag{2.3}$$

By using (1.3), we can write (2.3) as

$$\frac{1}{z}\left[f * \frac{xz + \left(x + \frac{[2]_q(1+Ax)}{B-A}\right)qz^2 + \frac{(1+q-[2]_q)(1+Ax)}{B-A}qz^3}{(1-z)(1-qz)(1-q^2z)}\right] \neq 0$$

which completes the proof. □

As $q \to 1^-$, we have following result proved by Ganesan et al. in [3].

Corollary 2.2. *The function $f \in C(A,B)$ in $|z| < R \leq 1$ if and only if*

$$\frac{1}{z}\left[f * \frac{xz + \frac{(Ax+Bx+2)}{B-A}z^2}{(1-z)^3}\right] \neq 0.$$

Remark 2.3. *As $q \to 1^-$ and $A = 1,\ B = -1$, we get convolution condition characterizing convex functions as in Silverman et al. in* [9] *with a suitable modification.*

Theorem 2.4. *The function $f \in S_q^*(A,B)$ in $|z| < R \leq 1$ if and only if*

$$\frac{1}{z}\left[f * \frac{xz + \frac{1+Ax}{B-A}qz^2}{(1-z)(1-qz)}\right] \neq 0,\ (|z| < R,\ |x| = 1).$$

Proof. Since $f \in S_q^*(A,B)$ if and only if $g(z) = \int_0^z \frac{f(\zeta)}{\zeta} d_q\zeta \in C_q(A,B)$, we have

$$\frac{1}{z}\left[g * \frac{xz + \left(x + \frac{[2]_q(1+Ax)}{B-A}\right)qz^2 + \frac{(1+q-[2]_q)(1+Ax)}{B-A}qz^3}{(1-z)(1-qz)(1-q^2z)}\right] = \frac{1}{z}\left[f * \frac{xz + \frac{1+Ax}{B-A}qz^2}{(1-z)(1-qz)}\right].$$

Thus the result follows from Theorem 2.1. □

As $q \to 1^-$, we have following result proved by Ganesan et al. in [3].

Corollary 2.5. *The function $f \in S^*(A,B)$ in $|z| < R \leq 1$ if and only if*

$$\frac{1}{z}\left[f * \frac{xz + \frac{1+Ax}{B-A}z^2}{(1-z)^2}\right] \neq 0,\ (|z| < R,\ |x| = 1).$$

As a corollary we can derive coefficient inequalities for the class $S_q^*(A,B)$.

Corollary 2.6. *A function $f \in \mathcal{A}$ is in the class $S_q^*(A,B)$ if and only if*

$$f(z) = 1 + \sum_{n=2}^{\infty} A_n z^{n-1} \neq 0,$$

where $A_n = \frac{([n]_1)+([n]_qB-A)x}{x(B-A)}a_n.$

Proof. A function $f \in S_q^*(A,B)$ if and only if
$$\frac{zD_qf(z)}{f(z)} \neq \frac{1+Ax}{1+Bx}.$$
That is
$$(1+Bx)(zD_qf(z)) - (1+Ax)f(z) \neq 0$$
which implies
$$(B-A)xz\left[1+\sum_{n=2}^{\infty}\left([n]_q(1+Bx)-(1+Ax)\right)a_nz^n\right] \neq 0.$$
This simplifies into
$$1+\sum_{n=2}^{\infty}\frac{([n]_1)([n]_qB-A)x}{x(B-A)}a_nz^{n-1} \neq 0,$$
which completes the proof. □

Remark 2.7. *As $q \to 1^-$ and $A = 1,\ B = -1$, we get convolution condition characterizing starlike functions as in Silverman et al. in* [9] *with a suitable modification.*

Now by using the concept of q-derivative we define the classes of Janowski q-spirallike and Janowski convex q-spirallike functions as the following

Definition 2.8. *A function $f \in \mathcal{A}$ is said to be in $S_q^{*\lambda}(A,B)$ if and only if*
$$e^{i\lambda}\frac{zD_qf(z)}{f(z)} \in P(A,B),\ |z| < R \leq 1,\ \lambda \text{ real with } |\lambda| < \frac{\pi}{2}.$$

Definition 2.9. *A function $f \in \mathcal{A}$ is said to be in $C_q^{\lambda}(A,B)$ if and only if*
$$e^{i\lambda}\frac{D_q(zD_qf(z))}{D_qf(z)} \in P(A,B),\ |z| < R \leq 1,\ \lambda \text{ real with } |\lambda| < \frac{\pi}{2}.$$

Theorem 2.10. *For $|z| < R \leq 1$, λ real with $|\lambda| < \frac{\pi}{2}$ and $|x| = 1$, we have*
$$e^{i\lambda}\frac{D_q(zD_qf(z))}{D_qf(z)} \in P(A,B)$$
if and only if
$$\frac{1}{z}\left[f * \frac{xz+\left(x+\frac{[2]_q(1+xe^{i\lambda}(A\cos\lambda+iB\sin\lambda))}{B-e^{-i\lambda}(A\cos\lambda+iB\sin\lambda)}\right)qz^2+\frac{(1+q-[2]_q)(1+xe^{i\lambda}(A\cos\lambda+iB\sin\lambda))}{B-e^{-i\lambda}(A\cos\lambda+iB\sin\lambda)}qz^3}{(1-z)(1-qz)(1-q^2z)}\right] \neq 0.$$

Proof. We have, $e^{i\lambda}\frac{D_q(zD_qf(z))}{D_qf(z)} \in P(A,B)$ if and only if

$$\frac{e^{i\lambda}\frac{D_q(zD_qf)}{D_qf} - i\sin\lambda}{\cos\lambda} \neq \frac{1+Ax}{1+Bx}, \ (|z|<R, \ |x|=1, \ x\neq -1)$$

which implies

$$(1+Bx)D_q\left(zD_qf\right) - \left[1 + xe^{i\lambda}(A\cos\lambda + iB\sin\lambda)\right]D_qf \neq 0. \tag{2.4}$$

Setting $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, we have

$$D_q\left(zD_qf\right) = 1 + \sum_{n=2}^{\infty}[n]_q^2 a_n z^{n-1} = D_qf * \frac{1}{(1-z)(1-qz)}.$$

The left hand side of (2.4) is equivalent to

$$(1+Bx)\left[D_qf * \sum_{n=1}^{\infty}[n]_q z^{n-1}\right] - D_qf * \sum_{n=1}^{\infty}\left[1 + xe^{i\lambda}(A\cos\lambda + iB\sin\lambda)\right]z^{n-1}$$

$$= D_qf * \sum_{n=1}^{\infty}\left[(1 + xe^{i\lambda}(A\cos\lambda + iB\sin\lambda)) + (1+Bx)[n]_q\right]z^{n-1}$$

$$= D_qf * \left(-\frac{\left[1 + xe^{i\lambda}(A\cos\lambda + iB\sin\lambda)\right]}{1-z} + \frac{1+Bx}{(1-z)(1-qz)}\right)$$

$$= D_qf * \left(\frac{\left[B - e^{i\lambda}(A\cos\lambda + iB\sin\lambda)\right]x + \left[1 + xe^{i\lambda}(A\cos\lambda + iB\sin\lambda)\right]qz}{(1-z)(1-qz)}\right).$$

Thus

$$\frac{1}{z}\left[zD_qf * \left(\frac{\left[B - e^{i\lambda}(A\cos\lambda + iB\sin\lambda)\right]xz + \left[1 + xe^{i\lambda}(A\cos\lambda + iB\sin\lambda)\right]qz^2}{(1-z)(1-qz)}\right)\right] \neq 0. \tag{2.5}$$

By using (1.3), we can write (2.13) as

$$\frac{1}{z}\left[f * \frac{xz + \left(x + \frac{[2]_q(1+xe^{i\lambda}(A\cos\lambda+iB\sin\lambda))}{B-e^{-i\lambda}(A\cos\lambda+iB\sin\lambda)}\right)qz^2 + \frac{(1+q-[2]_q)(1+xe^{i\lambda}(A\cos\lambda+iB\sin\lambda))}{B-e^{-i\lambda}(A\cos\lambda+iB\sin\lambda)}qz^3}{(1-z)(1-qz)(1-q^2z)}\right] \neq 0$$

which completes the proof. □

As $q \to 1^-$, we have following related result proved by Ganesan et al. in [3].

Corollary 2.11. *For $|z| < R \leq 1$, λ real with $|\lambda| < \frac{\pi}{2}$ and $|x| = 1$, we have*

$$e^{i\lambda}\frac{(zf'(z))'}{f'(z)} \in P(A,B)$$

if and only if

$$\frac{1}{z}\left[f * \frac{xz + \left(x + \frac{2(1+xe^{i\lambda}(A\cos\lambda+iB\sin\lambda))}{B-e^{-i\lambda}(A\cos\lambda+iB\sin\lambda)}\right)z^2}{(1-z)^3}\right] \neq 0.$$

Theorem 2.12. *For $|z| < R \leq 1$, λ real with $|\lambda| < \frac{\pi}{2}$ and $|x| = 1$, we have*

$$e^{i\lambda}\frac{zD_q f(z)}{f(z)} \in P(A,B)$$

if and only if

$$\frac{1}{z}\left[f * \left(\frac{\left[B - e^{i\lambda}(A\cos\lambda + iB\sin\lambda)\right]xz + \left[1 + xe^{i\lambda}(A\cos\lambda + iB\sin\lambda)\right]qz^2}{(1-z)(1-qz)}\right)\right] \neq 0.$$

Proof. The result follows from Theorem 2.10 in the same manner that Theorem 2.4 followed from Theorem 2.1. □

As $q \to 1^-$, we have following result proved by Ganesan et al. in [3].

Corollary 2.13. *For $|z| < R \leq 1$, λ real with $|\lambda| < \frac{\pi}{2}$ and $|x| = 1$, we have*

$$e^{i\lambda}\frac{zf'(z)}{f(z)} \in P(A,B)$$

if and only if

$$\frac{1}{z}\left[f * \left(\frac{\left[B - e^{i\lambda}(A\cos\lambda + iB\sin\lambda)\right]xz + \left[1 + xe^{i\lambda}(A\cos\lambda + iB\sin\lambda)\right]z^2}{(1-z)^2}\right)\right] \neq 0.$$

As a corollary we can derive coefficient inequalities for the class $S_q^{*\lambda}(A,B)$.

Corollary 2.14. *A function $f \in \mathcal{A}$ is in the class $S_q^{*\lambda}(A,B)$ if and only if*

$$f(z) = 1 + \sum_{n=2}^{\infty} d_n z^{n-1} \neq 0,$$

where $d_n = \frac{([n]_q-1)+([n]_qB-\gamma)x}{x(B-A)}a_n$ and $\gamma = (A\cos\lambda + iB\sin\lambda)e^{-i\lambda}$.

Proof. A function $f \in S^*_\lambda(A, B)$ if and only if

$$\frac{e^{i\lambda}\frac{zD_q f(z)}{f(z)} - i\sin\lambda}{\cos\lambda} \neq \frac{1+Ax}{1+Bx}.$$

That is

$$(1+Bx)(zD_q f(z)) - (1+\gamma x)f(z) \neq 0.$$

The rest of the proof follows as in Corollary 2.6. □

Remark 2.15. *As $q \to 1^-$ and $A = 1,\ B = -1$, we get convolution condition characterizing spirallikeness of functions as in Silverman et al. in* [9].

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