VERTEX-DEGREE BASED ECCENTRIC TOPOLOGICAL DESCRIPTORS OF ZERO DIVISOR GRAPHS OF FINITE COMMUTATIVE RINGS

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ABSTRACT. A graph $G(R)$ is said to be a zero divisor graph of commutative ring R with identity if $x_1, x_2 \in V(G(R))$ and $(x_1, x_2) \in E(G(R))$ if and only if $x_1.x_2 = 0$. The vertex-degree based eccentric topological indices of zero divisor graphs of commutative rings $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$ are studied in this paper, with p and q are primes.

1. INTRODUCTION

A single number that can be utilized to depict a few properties of the graph of a molecule is known as a topological descriptor for that graph. There are different topological descriptors that have found a number of applications in theoretical science. Topological descriptors are numerical parameters of a graph which characterize its topology and are usually graph invariant. Topological descriptors are utilized within the improvement of quantitative structure-activity connections (QSARs) and quantitative structure-property connections (QSPR) in which the organic movement or other properties of atoms are connected with their chemical structure [11, 12].

Algebraic structures have been studied for their close affiliation with representation theory and number theory, and they have been extensively studied in combinatorics [2, 13]. In addition to the extensive theoretical research in these areas, finite rings and fields have received attention for their applications to cryptography and coding theory.

A foundational role of molecular descriptors is to consider molecules as real bodies and transform them into numbers, which enables mathematics to play a key role in chemistry, pharmaceutical sciences, environmental protection, quality control and health research [24]. These molecular descriptors are graph invariants and are usually known as topological indices. In graph theory, by using topological indices we correlate various characteristics of a chemical structure in order to characterize it. These days there is an area of research devoted to computing topological indices for various structures.

These days, there exist a variety of topological descriptors that found a few applications in chemistry, physics, robotics, statistics, computer networks, etc. They can be classified based on the structural properties of graphs utilized for their calculation. The topological descriptors deals with the distance among the vertices in a graph are

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"distance based topological indices." Other topological descriptors that deals with the degree of vertices in graph are "degree based topological indices." The Wiener index is the first distance based topological index and it has eminent applications in chemistry. Wiener index is based on topological distance of vertices within the individual graph, the Hosoya index is calculated by checking of non-incident edges in a graph, the energy and the Estrada index are based on the range of the graph, the Randic connectivity index is calculated utilizing the degrees of vertices, etc. Zagreb indices, atom-bond connectivity, geometric-arithmetic index, for detail see [1, 9, 10, 21, 26].

2. Definitions and Notations

If G is a connected graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. A numerical quantity related to a graph that is invariant under graph automorphisms is topological index or topological descriptor. For a graph G, the degree of a vertex v is the number of edges incident with v and denoted by $d(v)$. The maximum degree of a graph G, denoted by $\Delta(G)$, and the minimum degree of a graph, denoted by $\delta(G)$, are the maximum and minimum degree of its vertices. The sum of degree of all vertices u which are adjacent to vertex v is denoted by $S(v)$. The distance between the vertices u and v of G is denoted by $d(u,v)$ and it is defined as the number of edges in a minimal path connecting them. Connectivity descriptors are important among topological descriptors and used in various fields like chemistry, physics and statistics. We are defining eccentricity of a vertex as opposed to the eccentric connectivity index.

$$e(u) = max\{d(u,v) : \forall\ v \in V(G)\} \tag{2.1}$$

In 1997 eccentric connectivity index was introduced by Sharma [27]. By using eccentric connectivity index, the mathematical modeling of biological activities of diverse nature is done. The general formula of eccentric connectivity index is defined:

$$\xi(G) = \sum_{v \in V(G)} d(v)\, e(v) \tag{2.2}$$

where $e(v)$ is the eccentricity of vertex v in G. Some applications and mathematical properties of eccentric connectivity index can be found in [23, 28]. The total eccentricity index is the sum of eccentricity of the all the vertex v in G. Total eccentricity index is introduced by Farooq and Malik [19], which is defined as:

$$\zeta(G) = \sum_{v \in V(G)} e(v) \tag{2.3}$$

The first Zagreb index of a graph G is studied in [25] based on degree and the new version of the first Zagreb index based on eccentricity was introduced by Ghorbani and Hosseinzadeh [20], as follows:

$$M_1{}^*(G) = \sum_{v \in V(G)} e(v)^2 \tag{2.4}$$

The eccentric connectivity polynomial is the polynomial version of the eccentric-connectivity index which was proposed by Alaeiyan, Mojarad and Asadpour [4] and some graph operations can be found in [8]. The eccentric connectivity polynomial of a graph G is defined as:

$$ECP(G,x) = \sum_{v \in V(G)} d(v) x^{e(v)} \tag{2.5}$$

Gupta, Singh and Madan [22] defined the augmented eccentric connectivity index of a graph G as follows:

$$\xi^{ac}(G) = \sum_{v \in V(G)} \frac{M(v)}{e(v)}, \tag{2.6}$$

where $M(v)$ denotes the product of degrees of all vertices u which are adjacent to vertex v. Some interesting results on augmented eccentric connectivity index are discussed in [14, 16]. Another very relevant and special eccentricity based topological index is connective eccentric index. The connective eccentric index was defined by Gupta, Singh and Madan [22] defined as: follows :

$$\xi^{C}(G) = \sum_{v \in V(G)} \frac{d(v)}{e(v)} \tag{2.7}$$

Ediz [17, 18] introduced the Ediz eccentric connectivity index and reverse eccentric connectivity index of graph G, which is used in various branches of sciences, molecular science and chemistry etc. Ediz eccentric connectivity index and reverse eccentric connectivity index defined as

$$E\zeta(G) = \sum_{v \in V(G)} \frac{S(v)}{e(v)} \tag{2.8}$$

$${}^{Re}\zeta(G) = \sum_{v \in V(G)} \frac{e(v)}{S(v)} \tag{2.9}$$

Let R be a commutative ring with identity and $Z(R)$ is the set of all zero divisors of R. $G(R)$ is said to be a zero divisor graph if $x, y \in V(G(R)) = Z(R)$ and $(x,y) \in E(G(R))$ if and only if $x.y = 0$. Beck [13] introduced the notion of zero divisor graph. Anderson and Livingston [7] proved that $G(R)$ is always connected if R is commutative. Anderson and Badawi [5] introduced the total graph of R as: there is an edge between any two distinct vertices $u, v \in R$ if and only if $u + v \in Z(R)$. For a graph G, the concept of graph parameters have always a high interest. Numerous authors briefly studied the zero-divisor and total graphs from commutative rings [3, 6].

Let p and q are prime numbers and $G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2})$ be zero divisor graph of the commutative rings $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$. In this paper, we investigate the eccentric topological descriptors namely, eccentric connectivity index, total eccentric index, first Zagreb eccentricity index, augmented eccentric connectivity index, connective eccentric index, Ediz eccentric index, of zero divisor graphs $G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2})$.

3. Results and Discussions

For any two prime numbers p and q, we consider the ring $R = \mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$ with usual addition and multiplication. The zero divisor graph $G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2})$ associated with ring R is defined as: Any $(x, y) \in V(G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}))$ if and only if the elements $x \in \{0, p, 2p, \ldots, (p-1)p\}$ or $y \in \{0, q, 2q, \ldots, (q-1)q\}$ with $(x, y) \neq (0, 0)$. Therefore, a non zero element (x, y) is not a zero divisor in the ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$ if and only if x is not a multiple of p and y is not a multiple of q. As the number of elements in $\mathbb{Z}_{p^2}$ which are not a multiple of p are $p^2 - p$ and the number of elements in $\mathbb{Z}_{q^2}$ which are not a multiple of q are $q^2 - q$. Also, the element $(0, 0)$ is not a zero divisor in $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$. Thus the total number of non zero divisor in the ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$ are $(p^2 - p)(q^2 - q) + 1$. This implies,

$$|V(G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}))| = p^2q^2 - \Big((p^2 - p)(q^2 - q) + 1\Big) = p^2q + pq^2 - pq - 1.$$

In order to discuss the degree of each vertex $(x, y) \in V(G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}))$, we have to see four different cases.

Case 1: If $x = 0$ and y is not a multiple of q, then each such vertex $(0, y)$ is only adjacent to the vertices $(x', 0)$ for every $x' \in \mathbb{Z}_{p^2} \setminus \{0\}$. Hence the degree of each vertex $(0, y)$ is $p^2 - 1$. By symmetry the degree of each vertex $(x, 0)$ such that x is not a multiple of p is $q^2 - 1$.

Case 2: If $x = 0$ and $y \in \{q, 2q, \ldots, (q-1)q\}$, then each such vertex $(0, y)$ is only adjacent to the vertices (x', y') for every $x' \in \mathbb{Z}_{p^2}$ and $y' \in \{0, q, 2q, \ldots, (q-1)q\}$ with $(x', y') \notin \{(0, 0), (0, y)\}$. Hence the degree of each vertex $(0, y)$ is $p^2q - 2$. By symmetry the degree of each vertex $(x, 0)$ such that $x \in \{p, 2p, \ldots, (p-1)p\}$ is $pq^2 - 2$.

Case 3: If x is not a multiple of p and $y \in \{q, 2q, \ldots, (q-1)q\}$, then each such vertex (x, y) is only adjacent to the vertices $(0, y')$ for every $y' \in \{q, 2q, \ldots, (q-1)q\}$. Hence the degree of each vertex (x, y) is $q - 1$. By symmetry the degree of each vertex (x, y) if $x \in \{p, 2p, \ldots, (p-1)p\}$ and y is not a multiple of q is $p - 1$.

Case 4: If $x \in \{p, 2p, \ldots, (p-1)p\}$ and $y \in \{q, 2q, \ldots, (q-1)q\}$, then each such vertex (x, y) is only adjacent to the vertices (x', y') for every $x' \in \{0, p, 2p, \ldots, (p-1)p\}$ and $y' \in \{0, q, 2q, \ldots, (q-1)q\}$ with $(x', y') \notin \{(0, 0), (0, y)\}$. Hence the degree of each vertex (x, y) is $pq - 2$.

Let us divide the vertices of $G(\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2})$ into seven sets corresponding to their degrees

$$V_i = \{u \in V(G(\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2})) : d(u) = i\},$$

$i = p^2-1, q^2-1, p^2q-2, pq^2-2, p-1, q-1, pq-2$. So, V_i contains the vertices of degree i. Note that $V(G(\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2})) = V_{p^2-1}\cup V_{q^2-1}\cup V_{p^2q-2}\cup V_{pq^2-2}\cup V_{p-1}\cup V_{q-1}\cup V_{pq-2}$. We get

(3.1)

$$|V_{p^2-1}| = q^2-q, |V_{q^2-1}| = p^2-p, |V_{p^2q-2}| = q-1, |V_{pq^2-2}| = p-1, |V_{p-1}| = q(p-1)(q-1)|,$$
$$|V_{q-1}| = p(p-1)(q-1), |V_{pq-2}| = (p-1)(q-1)$$

For our convenience, we use the following notations throughout the paper:

- If $x = kp$, then $1 \le k \le p-1$.
- If $x \neq kp$, then $0 \le k \le p-1$.
- If $y = tq$, then $1 \le t \le q-1$.
- If $y \neq tq$, then $0 \le t \le q-1$.

From the above discussion, we obtain the eccentricity of the vertices of $G(\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2})$ in the following lemma:

Lemma 3.1. *Let p, q be prime numbers. If $G(\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2})$ be the zero divisor graph of the ring $\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2}$, then the eccentricity of the vertices of $G(\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2})$ are 2 or 3, in particular $e(V_{p^2q-2}) = e(V_{pq^2-2}) = e(V_{pq-2}) = 2$ and $e(V_{p^2-1}) = e(V_{q^2-1}) = e(V_{p-1}) = e(V_{q-1}) = 3$.*

Proof. Let $(x,y) \in \mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2}$. By the definition of zero divisor graph and the above discussion, we observe that the vertices of type $(0,y), y \neq tq$ are adjacent with the vertices of type $(x,0), x \neq 0$. The vertices of type $(x,y), x = kp, y \neq tq$ are adjacent with the vertices of type $(x,0), x = kp$. The vertices of type $(x,y), x \neq kp, y = tq$ are adjacent with the vertices of type $(0,y), y = tq$. The vertices of type $(x,y), x = kp, y = tq$ are adjacent with the vertices of type $(x,0), x = kp$; $(0,y), y = tq$ and $(x',y'), x' = kp, y' = tq, x \neq x', y \neq y'$. The vertices of type $(x,0), x \neq kp$, are adjacent with the vertices of type $(0,y), y \neq 0$. The vertices of type $(0,y), y = tq$ are adjacent with the vertices of type $(x,0), x \neq 0$; $(x,y), x = kp, y = tq$; $(x,y), x \neq kp, y = tq$ and $(0,y'), y' = tq, y \neq y'$. The vertices of type $(x,0), x = kp$ are adjacent with the vertices of type $(0,y), y \neq 0$; $(x,y), x = kp, y \neq tq$; $(x,y), x \neq kp, y = tq$ and $(x',0), x' = kp, x \neq x'$. It is easy to see that the maximum distance of the vertices of the types $(0,y), y \neq tq$; $(x,0), x \neq kp$; $(x,y), x \neq kp, y = tq$; $(x,y), x = kp, y \neq tq$ in the zero divisor graph $G(\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2})$ of the ring $\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2}$ is 3 and the maximum distance of remaining vertices of $G(\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2})$ is 2. This completes the proof. □

The Table 1 is the summary of the partition sets with their degree, eccentricity and frequency. In the following theorem, we determine the eccentric connectivity index of the graph $G(\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2})$.

TABLE 1. The partition sets of vertices, their degree, eccentricity and frequency of the vertices in $G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2})$

Representation of of vertices	Degree	Eccentricity	Frequency
$(0,y), y \neq tq$	p^2-1	3	q^2-q
$(x,0), x \neq kp$	q^2-1	3	p^2-p
$(0,y), y = tq$	p^2q-2	2	$q-1$
$(x,0), x = kp$	pq^2-2	2	$p-1$
$(x,y), x = kp, y \neq tq$	$p-1$	3	$q(p-1)(q-1)$
$(x,y), x \neq kp, y = tq$	$q-1$	3	$p(p-1)(q-1)$
$(x,y), x = kp, y = tq$	$pq-2$	2	$(p-1)(q-1)$

Theorem 3.2. *Let p, q be prime numbers. If $G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2})$ be the zero divisor graph of the commutative ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$, then the eccentric connectivity index of $G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2})$ is*
$\xi(G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2})) = (p-1)(q-1)(14pq-4) + 2(2p^2q^2 - p^2q - pq^2 - 2p - 2q + 4).$

Proof. Using the values from Table 1 in the (2.2), we obtain

$$\begin{aligned}\xi(G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2})) = &\sum_{v \in V(G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}))} d(v)\, e(v)\\ =&3(p^2-1)(q^2-q) + 3(q^2-1)(p^2-p) + 2(q-1)(p^2q-2)\\ &+2(p-1)(pq^2-2) + 3q(p-1)^2(q-1) + 3p(p-1)(q-1)^2\\ &+2(p-1)(q-1)(pq-2)\end{aligned}$$

After simplification, we get
$\xi(G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2})) = (p-1)(q-1)(14pq-4) + 2(2p^2q^2 - p^2q - pq^2 - 2p - 2q + 4).$
This completes the proof. □

By using the Lemma 3.1 and Table 1 in (2.3) and equation (2.4), we obtain the total-eccentricity index and first eccentricity Zagreb index for $G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2})$ in the following corollaries.

Corollary 3.3. *Let p, q be prime numbers. If $G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2})$ be the zero divisor graph of the commutative ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$, then the total-eccentricity index of $G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2})$ is given by*
$\zeta(G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2})) = (p-1)(q-1)(3p+3q+2) + (p-1)(3p+2) + (q-1)(3q+2).$

Corollary 3.4. *Let p, q be prime numbers. If $G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2})$ be the zero divisor graph of the commutative ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$, then the first eccentricity Zagreb index of $G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2})$ is given by*
$M_1^*(G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2})) = (p-1)(q-1)(9p+9q+4) + (p-1)(9p+4) + (q-1)(9q+4).$

Theorem 3.5. *Let p, q be prime numbers. If $G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2})$ be the zero divisor graph of the commutative ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$, then the connective eccentric index of graph $G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2})$ is* $\xi^C(G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2})) = \frac{3p^2q^2-2p^2q-2pq^2-pq+2}{2} + \frac{4pq(p-1)(q-1)}{3}$.

Proof. By using the values of degrees and their eccentricity from Table 1 in the (2.7), we obtain:

$$\begin{aligned}\xi^c(G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2})) = & \sum_{v \in V(G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}))} \frac{d(v)}{e(v)} \\ = & \frac{(p-1)(pq^2-2)}{2} + \frac{(q-1)(p^2q-2)}{2} + \frac{(p-1)(q-1)(pq-2)}{2} \\ & + \frac{(p^2-1)(q^2-q)}{3} + \frac{(q^2-1)(p^2-p)}{3} + \frac{p(p-1)(q-1)^2}{3} \\ & + \frac{q(p-1)^2(q-1)}{3}\end{aligned}$$

After simplification, we get $\xi^C(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)) = \frac{3p^2q^2-2p^2q-2pq^2-pq+2}{2} + \frac{4pq(p-1)(q-1)}{3}$. This completes the proof. □

Theorem 3.6. *Let p, q be prime numbers. If $G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2})$ be the zero divisor graph of the commutative ring $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$, then the augmented eccentric connectivity index of $G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2})$ is* $\xi^{ac}(G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2})) =$

$$\frac{\alpha^{q-2}\beta^{p-2}\gamma^{pq-p-q}\left(\alpha\gamma(p+1)^{\alpha_1}(p-1)^{\beta-\gamma+5}+\beta\gamma(q+1)^{\beta_1}(q-1)^{\alpha-\gamma+5}+\alpha\beta(pq-p-q+1)\right)}{2}$$
$$+\frac{\beta_1\alpha^{q-1}\left((p^2-1)^{\alpha_1}+q-1\right)+\alpha_1\beta^{p-1}\left((q^2-1)^{\beta_1}+p-1\right)}{3},$$

where $\alpha = p^2q-2, \beta = pq^2-2, \gamma = pq-2, \theta = q^2-q$ and $\lambda = p^2-p$.

Proof. Let $M(v)$ be the product of degrees of all vertices which are at distance one with vertex v. By putting the values in (2.6), which are obtained from the proof of Lemma 3.1 and Table 1, we obtain

$$\xi^{ac}(G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2})) = \sum_{v \in V(G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}))} \frac{M(v)}{e(v)}$$

$$= \frac{(p-1)(p^2-1)^{q^2-q}(p^2q-2)^{q-1}(pq-2)^{(p-1)(q-1)}(p-1)^{q(p-1)(q-1)}(pq^2-2)^{p-2}}{2}$$
$$+\frac{(q-1)(q^2-1)^{p^2-p}(pq^2-2)^{p-1}(pq-2)^{(p-1)(q-1)}(q-1)^{p(p-1)(q-1)}(p^2q-2)^{q-2}}{2}$$
$$+\frac{(p-1)(q-1)(p^2q-2)^{q-1}(pq-2)^{(p-1)(q-1)-1}(pq^2-2)^{p-1}}{2}$$

$$+\frac{(q^2-q)(pq^2-2)^{p-1}(q^2-1)^{p^2-p}}{3}+\frac{(p^2-p)(p^2q-2)^{q-1}(p^2-1)^{q^2-q}}{3}$$
$$+\frac{q(p-1)(q-1)(pq^2-2)^{p-1}}{3}+\frac{p(p-1)(q-1)(p^2q-2)^{q-1}}{3}.$$

After simplification, we obtain $\xi^{ac}(G(\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2}))=$

$$\frac{\alpha^{q-2}\beta^{p-2}\gamma^{pq-p-q}\left(\alpha\gamma(p+1)^{\theta}(p-1)^{\beta-\gamma+5}+\beta\gamma(q+1)^{\lambda}(q-1)^{\alpha-\gamma+5}+\alpha\beta(pq-p-q+1)\right)}{2}$$
$$+\frac{\lambda\alpha^{q-1}\left((p^2-1)^{\theta}+q-1\right)+\theta\beta^{p-1}\left((q^2-1)^{\lambda}+p-1\right)}{3},$$

with $\alpha=p^2q-2,\beta=pq^2-2,\gamma=pq-2,\theta=q^2-q$ and $\lambda=p^2-p$.

This completes the proof. □

Theorem 3.7. *Let p,q be prime numbers. If $G(\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2})$ be the zero divisor graph of the commutative ring $\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2}$, then the Ediz eccentric connectivity index of graph $G(\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2})$ is $E\zeta(G(\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2}))=$*
$\frac{3p^3q^3-2p^3q-12p^2q^2+10p^2q-2pq^3+10pq^2-4}{2}+\frac{(p-1)(q-1)(p^3q^2+p^3q+p^2q^3+pq^3-6pq)}{3}$.

Proof. $S(v)$ is the sum of degrees of all vertices u which are adjacent to vertex v. Calculate the values of $S(v)$ with the help of Table 1. Also the eccentricity of each vertex is given in the Table 1. Putting these values in (2.8), we obtain: $E\zeta(G(\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2}))=\sum\limits_{v\in V(G(\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2}))}\frac{S(v)}{e(v)}$

$=(p-1)\frac{(p^2-1)(q^2-q)+(p^2q-2)(q-1)+(pq-2)(p-1)(q-1)+q(p-1)(p-1)(q-1)+(pq^2-2)(p-2)}{2}$
$+(q-1)\frac{(q^2-1)(p^2-p)+(pq^2-2)(p-1)+(pq-2)(p-1)(q-1)+p(q-1)(p-1)(q-1)+(p^2q-2)(q-2)}{2}$
$+(p-1)(q-1)\frac{(p^2q-2)(q-1)+(pq-2)((p-1)(q-1)-1)+(pq^2-2)(p-1)}{2}$
$+(q^2-q)\frac{(pq^2-2)(p-1)+(q^2-1)(p^2-p)}{3}+(p^2-p)\frac{(p^2q-2)(q-1)+(p^2-1)(q^2-q)}{3}$
$+q(p-1)(q-1)\frac{(pq^2-2)(p-1)}{3}+p(p-1)(q-1)\frac{(p^2q-2)(q-1)}{3}$

After simplification we get $E\zeta(G(\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2}))=\frac{3p^3q^3-2p^3q-12p^2q^2+10p^2q-2pq^3+10pq^2-4}{2}+\frac{(p-1)(q-1)(p^3q^2+p^3q+p^2q^3+pq^3-6pq)}{3}$. This completes the proof. □

Theorem 3.8. *Let p,q be prime numbers. If $G(\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2})$ be the zero divisor graph of the commutative ring $\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2}$, then the eccentric connectivity polynomial of $G(\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2})$ is*
$ECP(G(\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2}),x)=$
$4(p^2q^2-p^2q-pq^2+pq)x^3+(3p^2q^2-2p^2q-2pq^2-pq+2)x^2$.

Proof. By using the degree and their corresponding eccentricity from the Table 1 in the (2.5), we obtain $ECP(G(\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2}),x)=\sum\limits_{v\in V(G(\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2}))}d(v)x^{e(v)}$

$= (p^2-1)(q^2-q)x^3 + (q^2-1)(p^2-p)x^3 + q(p-1)^2(q-1)x^3 + p(p-1)(q-1)^2x^3$
$+(p-1)(pq^2-2)x^2 + (q-1)(p^2q-2)x^2 + (p-1)(q-1)(pq-2)x^2$

After simplification, we obtain

$ECP(G(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}), x) = 4(p^2q^2 - p^2q - pq^2 + pq)x^3 + (3p^2q^2 - 2p^2q - 2pq^2 - pq + 2)x^2$. This completes the proof. □

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