On the c-dominating Estrada index of a graph

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Abstract

Let G be a graph with n vertices then the c-dominating matrix of G is the square matrix of order n whose (i, j)-entry is equal to 1 if the i-th and j-th vertex of G are adjacent, is also equal to 1 if the $i = j, v_i \in D_c$ and zero otherwise. The c-dominating energy of a graph G, is defined as the sum of the absolute values of all eigenvalues the c-dominating matrix. The main purposes of this paper are to introduce the c-dominating Estrada index of a graph. Moreover, obtain upper and lower bounds for the$c$-dominating Estrada index and investigate the relations between the c-dominating Estrada index and the c-dominating energy.

Keywords: Eigenvalue of graph, Energy, c-dominating Estrada index, c-dominating Energy, Connected dominating number.

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1 Introduction

Let G be such a graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex $v \in V(G)$ is the number of vertices adjacent to v and is denoted by $d_G(v)$.

Let the vertices of G be labeled as $v_1, v_2, \ldots, v_n$. The vertex u and v are adjacent if $uv \in E(G)$. The open(closed) neighborhood of a vertex $v \in V(G)$ is $N(v) = \{u : uv \in E(G)\}$ and $N[v] = N(v) \cup v$ respectively. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$ and is defined as $d_G(v) = |N(v)|$. A vertex $v \in V(G)$ is pendant if $|N(v)| = 1$ and is called support vertex if it is adjacent to pendant vertex. Let us denote the complete graph, and the path having n vertices by K_n and P_n, respectively The complement of G, denoted by $\overline{G}$, is a graph which has the same vertices as G, and in which two vertices are adjacent if and only if they are not adjacent in G.

Any vertex $v \in V(G)$ with $|N(v)| > 1$ is called internal vertex. A subset $D \subseteq V(G)$ is called dominating set if $N[D] = V(G)$. The minimum cardinality of such a set D is called the domination number $\gamma(G)$ of G. A dominating set D is connected if the subgraph induced by D is connected. The minimum cardinality of connected dominating set D is called the connected dominating number $\gamma_c(G)$ of G [28]. The *adjacency* matrix of a graph G is a square matrix $A(G) = [a_{ij}]$ of order n, defined via

$$a_{ij} = \begin{cases} 1 & \text{if the vertices } v_i \text{ and } v_j \text{ of } G \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

The *eigenvalues* of $A(G)$ are the adjacency eigenvalues of G, they are labeled as $\lambda_1, \lambda_2, \ldots, \lambda_n$. The graph energy is a graph-spectrum-based quantity, introduced in the 1970s. After a latent period of 20-30 years, it became a popular topic of research both in mathematical chemistry and in "pure" spectral graph theory, resulting in over 600 published papers. The first paper in which graph energy was defined as the sum of absolute values of the eigenvalues of the $(0;1)$-adjacency matrix, namely as

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i|. \tag{1}$$

The concept of energy first by *I. Gutman* be used to approximate the total *π-electron* energy of a *molecule* (see, e.g. [9, 10]). For more details on energy mathematical properties see, ([1, 3], [16]-[19]).

A graph-spectrum-based graph invariant, recently put forward by Estrada [7, 8], is defined as

$$EE = EE(G) = \sum_{i=1}^{n} e^{\lambda_i}.$$

EE is nowadays usually referred to as the Estrada index, see [26]. Although invented only a few years ago [7], the Estrada index has already found numerous applications. It was used to quantify the degree of folding of long-chain molecules, especially proteins [8]. Some mathematical properties of the Estrada index were established in [4, 13, 14, 15]. One of most important properties is the following:

$$EE = \sum_{i=1}^{\infty} \frac{M_k(G)}{k!}. \tag{2}$$

Denoting by $M_k = M_k(G)$ to the *k-th moment* of the graph G, we get where,$M_k = M_k(G) = \sum_{i=1}^{n}(\lambda_i)^k$. It is well known that $M_k(G)$ is equal to the number of closed walks of length k in G.

Refer to references [20, 21, 22, 23, 24, 27] to see recent work on the Eatrada index.

The c-dominating matrix of G denoted by $D_c(G) = (d_{ij})$, defined in [11] as following

$$d_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } v_i \in D_c, \\ 1 & \text{if the vertices } v_i \text{ and } v_j \text{ of } G \text{ are adjacent }, \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues of $D_c(G)$ labeled as $\theta_1 \geqslant \theta_2 \geqslant \cdots \geqslant \theta_n$ are said to be D_c*-eigenvalues* of G and their collection is called the c-dominating spectrum or D_c*-spectrum* of G. The spectral radius of G, denoted by $\theta_1(G)$, is the largest eigenvalue of $D_c(G)$.

In [11], the c-dominating energy of G is then defined as following

$$E_c = E_c(G) = \sum_{i=1}^{n} \mid \theta_i \mid .$$

Similarly to the definition of Estrada index, we define the c-dominating Estrada index of G, as following

$$EE_c = EE_c(G) = \sum_{i=1}^{n} e^{\theta_i}.$$

Now, here, we denoted

$$N_k = \sum_{i=1}^{n} (\theta_i)^k$$

hence, with an argument similar to (2), we can write that

$$EE_c(G) = \sum_{i=1}^{\infty} \frac{N_k}{k!}.$$

This paper is organized as follows. In Section 2, we give a list of some previously known results. In Section 3, introducing c-dominating Estrada index and we establish upper and lower bounds for it. In Section 4, we investigate the relations between c-dominating Estrada index and c-dominating energy. Note that all graphs considered in this paper are simple.

2 Preliminaries and known results

In this section, we shall list some previously known results that will be needed in the next sections. Now we give some lemmas which will be needed later.
The following result comes from [11].

Lemma 1. *[11] Let G be a graph with n vertices and c-dominating matrix D_c. Then*

$$(1) \quad N_1 = \sum_{i=1}^{n} \theta_i = tr(D_c) = \gamma_c(G) = \gamma_c, \tag{3}$$

$$(2) \quad N_2 = \sum_{i=1}^{n} (\theta_i)^2 = tr(D_c^2) = 2m + \gamma_c(G). \tag{4}$$

Now, we obtain upper bound for N_4 that will be need to obtain the result.

Lemma 2. *Let G be a graph with n vertices and $\theta_1 \geqslant \theta_2 \geqslant \cdots \geqslant \theta_n$ be a non-increasing arrangement of eigenvalues of G. Then, the following inequality is valid*

$$\sum_{i=1}^{n} \theta_i^4 \leqslant (\theta_n^2 + \theta_1^2)(2m + \gamma_c(G)) - n\theta_1^2\theta_n^2. \tag{5}$$

Proof. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers for which there exist real constants R and r, so that for each $i, i = 1, 2, \ldots, n$, holds $ra_i \leqslant b_i \leqslant Ra_i$. Then the following inequality is valid (see [5])

$$\sum_{i=1}^{n} b_i^2 + rR \sum_{i=1}^{n} a_i^2 \leqslant (r + R) \sum_{i=1}^{n} a_i b_i. \tag{6}$$

Equality in (6) holds if and only if for et least one $i, 1 \leqslant i \leqslant n$ holds $ra_i = bi = Ra_i$.

Setting $a_i := 1$, $b_i := \theta_i^2$, $r := \theta_n^2$ and $R := \theta_1^2$, $i = 1, 2, \ldots n$, in Inequality (6), we get

$$\sum_{i=1}^{n} \theta_i^4 + \theta_1^2 \theta_n^2 \sum_{i=1}^{n} 1 \leqslant (\theta_n^2 + \theta_1^2) \sum_{i=1}^{n} \theta_i^2.$$

By Equalitity (4), we have,

$$\sum_{i=1}^{n} \theta_i^2 = 2m + \gamma_c(G).$$

Also, we know that

$$\sum_{i=1}^{n} 1 = n,$$

Therefore, the above inequality becomes

$$\sum_{i=1}^{n} \theta_i^4 \leqslant (\theta_n^2 + \theta_1^2)(2m + \gamma_c(G)) - n\theta_1^2 \theta_n^2.$$

□

Remark 1. *For any real x, the power-series expansion of e^x, is the following*

$$e^x = \sum_{k \geqslant 0} \frac{x^k}{k!}. \tag{7}$$

Remark 2. *For nonnegative $x_1, x_2, \ldots, x_n$ and $k \geqslant 2$,*

$$\sum_{i=1}^{n} (x_i)^k \leqslant (\sum_{i=1}^{n} {x_i}^2)^{\frac{k}{2}}. \tag{8}$$

Lemma 3. *(Rayleigh-Ritz) [12] If B is a real symmetric $n \times n$ matrix with eigenvalues $\lambda_1(B) \geq \lambda_2(B) \geq \cdots \geq \lambda_n(B)$, then for any $X \in R^n$, $(X \neq 0)$,*

$$X^t B X \leqslant \lambda_1(B) X^t X.$$

Equality holds if and only if X is an eigenvector of B, corresponding to the largest eigenvalue $\lambda_1(B)$.

3 Bounds for the c-dominating Estrada index of graphs

In this section, we consider the c-dominating Estrada index of graph G. Also we present lower and upper bounds for the c-dominating Estrada index in terms of the number of vertices, edges and the connected dominating number.

At first, we state the following useful lemma.

Lemma 4. *Let G be a connected graph with $n \geqslant 2$ vertices. Then the spectral radius of the c-dominating matrix is bounded from below as*

$$\theta_1 \geqslant \frac{2D_c(G)}{n}. \tag{9}$$

Proof. Let $D_c = ||d_{ij}||$ be c-dominating matrix corresponding to D_c. By Lemma 3, for any vector $X = (x_1, x_2, \ldots, x_n)^t$,

$$\begin{aligned} X^t D_c X &= \left(\sum_{j,j\sim 1}^{n} x_j d_{j1}, \sum_{j,j\sim 2}^{n} x_j d_{j2}, \ldots, \sum_{j,j\sim n}^{n} x_j d_{jn} \right)^t X \\ &= 2\sum_{i\sim j} d_{ij} x_i x_j \end{aligned} \tag{10}$$

because $d_{ij} = d_{ji}$. Also,

$$X^t X = \sum_{i=1}^{n} x_i^2. \tag{11}$$

Using Eqs. (10) and (11), by Lemma 3, we obtain

$$\theta_1 \geqslant \frac{2\sum\limits_{i\sim j} d_{ij} x_i x_j}{\sum\limits_{i=1}^{n} x_i^2}. \tag{12}$$

Since (12) is true for any vector X, by putting $X = (1, 1, \ldots, 1)^t$, we have

$$\theta_1 \geqslant \frac{2D_c(G)}{n}.$$

$\square$

We are now ready to give some new bounds for $EE_c(G)$.

Theorem 1. *Let G be a graph with n vertices. Then*

$$EE_c(G) \geqslant e^{\frac{2D_c(G)}{n}} + \frac{n-1}{e^{\left(\frac{\frac{2D_c(G)-\gamma_c}{n}}{n-1}\right)}}.$$

Proof. By the definition of c-dominating Estrada index and using arithmetic-geometric mean inequality, we obtain

$$
\begin{aligned}
EE_c(G) &= e^{\theta_1} + e^{\theta_2} + \cdots + e^{\theta_n} \\
&\geqslant e^{\theta_1} + (n-1)\left(\prod_{i=2}^{n} e^{\theta_i}\right)^{\frac{1}{n-1}} && (13) \\
&= e^{\theta_1} + (n-1)\left(e^{\gamma_c - \theta_1}\right)^{\frac{1}{n-1}} \quad \text{by Equality (3).} && (14)
\end{aligned}
$$

Now we consider the following function

$$f(x) = e^x + \frac{n-1}{e^{\frac{x-\gamma_c}{n-1}}}$$

for $x > 0$. We have

$$f(x) \geqslant e^x + \frac{n-1}{e^{\frac{x-\gamma_c}{n-1}}}$$

It is easy to see that f is an increasing function for $x > 0$. From the Equation (14) and Lemma 4, we get

$$EE_c(G) \geqslant e^{\frac{2D_c(G)}{n}} + \frac{n-1}{e^{\left(\frac{\frac{2D_c(G)-\gamma_c}{n}}{n-1}\right)}}.$$

□

Theorem 2. *Let G be a graph with $n \geq 5$ vertices, $m \geq 5$ edges and $\gamma_c(G)$ the connected dominating number. Then*

$$EE_c(G) \leqslant n - 1 + e^{\sqrt{2m+\gamma_c(G)-1}}. \tag{15}$$

Proof. Let the number of positive eigenvalues of G be n_+. Since $f(x) = e^x$ monotonically increases in the interval $(-\infty, +\infty)$ and $m \neq 0$, we get

$$
\begin{aligned}
EE_c(G) = \sum_{i=1}^{n} e^{\theta_i} < n - n_+ + \sum_{i=1}^{n_+} e^{\theta_i} &= n - n_+ + \sum_{i=1}^{n_+} \sum_{k\geqslant 0} \frac{(\theta_i)^k}{k!} \\
&= n + \sum_{k\geqslant 1} \frac{1}{k!} \sum_{i=1}^{n_+} (\theta_i)^k && (16) \\
&\leqslant n + \sum_{k\geqslant 1} \frac{1}{k!} \left[\sum_{i=1}^{n_+} \theta_i^2\right]^{\frac{k}{2}} \\
&= n + \sum_{k\geqslant 1} \frac{1}{k!} \left[2m + \gamma_c(G) - \sum_{i=n_++1}^{n} \theta_i^2\right]^{\frac{k}{2}}.
\end{aligned}
$$

Since every graph with $n \geq 5$ vertices and $m \geq 5$ edges has P_5 as its induced subgraph and by direct calculation it can be easily seen that the spectrum of P_5 are $\theta_1 = 2.618, \theta_2 = 1.618, \theta_3 = 0.382, \theta_4 = -1.000, \theta_5 = -0.618$, it follows from the interlacing inequalities that $\theta_n \leqslant 1$, which implies that, $\sum_{i=n_+ +1}^{n} \theta_i^2 \geqslant 1$. Consequently,

$$EE_c(G) \leq n + \sum_{k \geqslant 1} \frac{1}{k!} \Big[2m + \gamma_c(G) - 1 \Big]^{\frac{k}{2}} = n - 1 + e^{\sqrt{2m+\gamma_c(G)-1}}.$$

□

Theorem 3. *Let G be a graph with n vertices, m edges and $\gamma_c(G)$ the connected dominating number. Then*

$$EE_c(G) \leq n - 1 + \gamma_c + \frac{2m+\gamma_c}{2} + \frac{N_3}{6} - N_4^{1/4} - \frac{N_4^{1/2}}{2} - \frac{N_4^{3/4}}{6} + e^{\sqrt[4]{2m+\gamma_c(G)-n\theta_1\theta_n}}.$$

Proof. By definition of the c-dominating Estrada index, we have

$$\begin{aligned}
EE_c(G) = \sum_{i=1}^{n} e^{\theta_i} = \sum_{i=1}^{n}\sum_{k=0}^{\infty} \frac{\theta_i^k}{k!} &= n + \sum_{i=1}^{n} \theta_i + \frac{1}{2}\sum_{i=1}^{n} \theta_i^2 + \frac{1}{6}\sum_{i=1}^{n} \theta_i^3 + \sum_{i=1}^{n}\sum_{k=4}^{\infty} \frac{\theta_i^k}{k!} \\
&= n + \sum_{i=1}^{n} \theta_i + \frac{1}{2}\sum_{i=1}^{n} \theta_i^2 + \frac{1}{6}\sum_{i=1}^{n} \theta_i^3 + \sum_{i=1}^{n}\sum_{k=4}^{\infty} \frac{|\theta_i|^k}{k!} \\
&= n + \sum_{i=1}^{n} \theta_i + \frac{1}{2}\sum_{i=1}^{n} \theta_i^2 + \frac{1}{6}\sum_{i=1}^{n} \theta_i^3 + \sum_{k=4}^{\infty} \frac{1}{k!} \sum_{i=1}^{n} \left(\theta_i^4\right)^{\frac{k}{4}} \\
&\leq n + \sum_{i=1}^{n} \theta_i + \frac{1}{2}\sum_{i=1}^{n} \theta_i^2 + \frac{1}{6}\sum_{i=1}^{n} \theta_i^3 + \sum_{k=4}^{\infty} \frac{1}{k!} \left(\sum_{i=1}^{n} \theta_i^4\right)^{\frac{k}{4}} \\
&= n - 1 + \sum_{i=1}^{n} \theta_i + \frac{1}{2} N_2 + \frac{1}{6} N_3 \\
&\quad - N_4^{1/4} - \frac{1}{2} N_4^{1/2} - \frac{1}{6} N_4^{3/4} + \sum_{k=0}^{\infty} \frac{\sqrt[4]{N_4^k}}{k!} \\
&= n - 1 + \gamma_c + \frac{2m+\gamma_c}{2} + \frac{N_3}{6} - N_4^{1/4} - \frac{N_4^{1/2}}{2} - \frac{N_4^{3/4}}{6} + e^{\sqrt[4]{N_4}}.
\end{aligned}$$

Therefore, by Lemma 5 and Lemma 2, we get

$$EE_c(G) \leq n - 1 + \gamma_c + \frac{2m+\gamma_c}{2} + \frac{N_3}{6} - N_4^{1/4} - \frac{N_4^{1/2}}{2} - \frac{N_4^{3/4}}{6} + e^{\sqrt[4]{(\theta_n^2+\theta_1^2)(2m+\gamma_c(G)) - n\theta_1^2\theta_n^2}}.$$

□

Theorem 4. Let G be a graph with n vertices and m edges. Then

$$EE_c(G) \geq \sqrt{n^2 + 4m + (n+4)\,\gamma_c + \frac{4N_3}{3}}\ .$$

Proof. For any real x, one has

$$e^x \geq 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}. \tag{17}$$

Since $e^x \geqslant 1 + x$, therefore, by this fact, Inequality (17) and the definition of the c-dominating Estrada index, we have

$$\begin{aligned}
EE(G)^2 &= (\textstyle\sum_{i=1}^n e^{\theta_i})^2 \\
&= (\textstyle\sum_{i=1}^n e^{\theta_i})(e^{\theta_1} + e^{\theta_n}) + (\sum_{i=1}^n e^{\theta_i})(e^{\theta_2} + \ldots + e^{\theta_{n-1}}) \\
&\geq \textstyle\sum_{i=1}^n e^{2\theta_i} + ne^{\theta_1+\theta_n} + (\sum_{i=1}^n e^{\theta_i})(e^{\theta_2} + \ldots + e^{\theta_{n-1}}) \\
&\geq \textstyle\sum_{i=1}^n (1 + 2\theta_i + 2\theta_i^2 + \frac{8}{6}\theta_i^3) + ne^{\theta_1+\theta_n} + (\sum_{i=1}^n e^{\theta_i})(e^{\theta_2} + \ldots + e^{\theta_{n-1}}) \\
&= n + 2\textstyle\sum_{i=1}^n \theta_i + 2\sum_{i=1}^n \theta_i^2 + \frac{8}{6}\sum_{i=1}^n \theta_i^3 + ne^{\theta_1+\theta_n} + (\sum_{i=1}^n e^{\theta_i})(e^{\theta_2} + \ldots + e^{\theta_{n-1}}) \\
&\geq n + 2\gamma_c + 2\,(2m + \gamma_c) + \tfrac{4N_3}{3} + n(1 + \theta_1 + \theta_n) + n((1 + \theta_2) + \ldots + (1 + \theta_{n-1})) \\
&\geq n^2 + 4m + (n+4)\,\gamma_c + \tfrac{4N_3}{3},
\end{aligned}$$

and this leads to the desired bound. □

4 Bound for the c-dominating Estrada index involving the c-dominating energy

In this section, we obtain lower and upper boundf for c-dominating Estrada index relates to c-dominating energy.

Theorem 5. *The c-dominating Estrada index $EE_c(G)$ and the c-dominating energy $E_c(G)$ satisfy the following inequality:*

$$EE_c(G) \geq \frac{1}{2}E_c(G)(e-1) + (n - n_+) + \gamma_c(G). \tag{18}$$

Proof. Since $e^x \geqslant 1 + x$, equality holds if and only if $x = 0$ and $e^x \geqslant ex$, equality holds if and only if $x = 1$, we have

$$\begin{aligned}
EE_c(G) &= \sum_{i=1}^{n} e^{\theta_i} = \sum_{\theta_i > 0} e^{\theta_i} + \sum_{\theta_i \leqslant 0} e^{\theta_i} \\
&\geqslant \sum_{\theta_i > 0} e\theta_i + \sum_{\theta_i \leqslant 0} (1 + \theta_i) \\
&= e(\theta_1 + \theta_2 + \cdots + \theta_{n_+}) + (n - n_+) + (\theta_{n_+ + 1} + \cdots + \theta_n) \\
&= (e-1)(\theta_1 + \theta_2 + \cdots + \theta_{n_+}) + (n - n_+) + \sum_{i=1}^{n} \theta_i \\
&= \frac{1}{2} E_c(G)(e-1) + (n - n_+) + \gamma_c(G).
\end{aligned}$$

□

Theorem 6. *Let G be a graph with largest eigenvalue θ_1 and let p, τ and q be, respectively, the number of positive, zero and negative eigenvalues of G. Then*

$$EE_c(G) \geqslant \tau + pe^{\frac{E_c(G)+\gamma_c}{2p}} + qe^{\frac{\gamma_c - E_c(G)}{2q}}. \tag{19}$$

Proof. Let $\theta_1 \geqslant \cdots \geqslant \theta_p$ be the positive, and $\theta_{n-q+1}, \ldots, \theta_n$ be the negative eigenvalues of G. By Equality (3), we have

$$\sum_{i=1}^{p} \theta_i + \sum_{i=n-q+1}^{n} \theta_i = \gamma_c. \tag{20}$$

And by the definition of the c-dominating energy, we have

$$E_c(G) = \sum_{i=1}^{p} \theta_i + \sum_{i=n-q+1}^{n} |\theta_i|. \tag{21}$$

By using Equalities (20) and (21), we have

$$\begin{aligned}
E_c(G) &= \sum_{i=1}^{p} \theta_i + \sum_{i=n-q+1}^{n} |\theta_i| \\
&= \gamma_c - \sum_{i=n-q+1}^{n} \theta_i + \sum_{i=n-q+1}^{n} |\theta_i| \\
&= \gamma_c + 2 \sum_{i=n-q+1}^{n} |\theta_i| \qquad (22) \\
&= \gamma_c - 2 \sum_{i=n-q+1}^{n} \theta_i.
\end{aligned}$$

Therefore, we get

$$\sum_{i=n-q+1}^{n} \theta_i = \frac{\gamma_c - E_c(G)}{2}. \tag{23}$$

With the same argument as above as well as using Enequality (22), we have

$$\sum_{i=p}^{n} \theta_i = \frac{E_c(G) + \gamma_c}{2}. \tag{24}$$

By the *arithmetic-geometric* mean inequality and Enequality (23), we have

$$\sum_{i=1}^{p} e^{\theta_i} \geqslant pe^{\frac{(\theta_1 + \cdots + \theta_p)}{p}} = pe^{\frac{E_c(G)+\gamma_c}{2p}}. \tag{25}$$

Similarly and by using Enequality (24), we can write

$$\sum_{i=n-q+1}^{n} e^{\theta_i} \geqslant qe^{\frac{\gamma_c - E_c(G)}{2q}}. \tag{26}$$

For the zero eigenvalues, we have

$$\sum_{i=p+1}^{n-q} e^{\theta_i} = \tau.$$

So we obtain

$$EE_c(G) \geqslant \tau + pe^{\frac{E_c(G)+\gamma_c}{2p}} + qe^{\frac{\gamma_c - E_c(G)}{2q}}.$$

□

Theorem 7. *Let G be a non-empty graph with n vertices and m edges. Then*

$$EE_c(G) - E_c(G) \leq n - 1 - \sqrt{2m + \gamma_c(G) - 1} + e^{\sqrt{2m+\gamma_c(G)-1}}.$$

Proof. By Inequality (16), we have

$$\begin{aligned}
EE_c(G) \leq n + \sum_{k\geqslant 1} \frac{1}{k!} \sum_{i=1}^{n_+} (\theta_i)^k &= n + \frac{E_c(G)}{2} + \sum_{i=1}^{n_+} \sum_{k\geqslant 2} \frac{(\theta_i)^k}{k!} \\
&< n + E_c(G) + \sum_{i=1}^{n_+} \sum_{k\geqslant 2} \frac{(\theta_i)^k}{k!} \\
&= n + E_c(G) + \sum_{k\geqslant 2} \frac{1}{k!} \left(\sum_{i=1}^{n_+} \theta_i \right)^k \\
&\leq n + E_c(G) + \sum_{k\geqslant 2} \frac{1}{k!} \left(\sum_{i=1}^{n_+} \theta_i^2 \right)^{\frac{k}{2}} \\
&\leq n + E_c(G) + \sum_{k\geqslant 2} \frac{1}{k!} \left(2m + \gamma_c - 1\right)^{\frac{k}{2}}.
\end{aligned}$$

Therefore, we have

$$EE_c(G) - E_c(G) \leq n - 1 - \sqrt{2m + \gamma_c(G) - 1} + e^{\sqrt{2m+\gamma_c(G)-1}}.$$

$\square$

Theorem 8. *Let G be a graph with n vertices. Then*

$$EE_c(G) \leqslant n - 1 + e^{E_c(G)},$$

and equality holds if and only if $G \cong \overline{K}_n$.

Proof. By definition of c-dominating Estrada index, we have

$$\begin{aligned} EE_c(G) &\leqslant n + \sum_{i=1}^{n} \sum_{k\geqslant 1} \frac{\mid \theta_i \mid^k}{k!} \\ &= n + \sum_{k\geqslant 1} \frac{1}{k!} \left(\sum_{i=1}^{n} \mid \theta_i \mid^k \right) \\ &\leqslant n + \sum_{k\geqslant 1} \frac{1}{k!} \left(\sum_{i=1}^{n} \mid \theta_i \mid \right)^k \\ &= n + \sum_{k\geqslant 1} \frac{(E_c(G))^k}{k!} \end{aligned}$$

which implies

$$EE_c(G) \leqslant n - 1 + e^{E_c(G)}.$$

It is not difficult to see that equality holds if and only if $G \cong \overline{K}_n$. $\square$

Concluding Remarks

In this paper, the c-dominating Estrada index of a graph is introduced and the c-dominating energy and the c-dominating Estrada index are studied. Moreover, we investigate the relations between the c-dominating Estrada index and the c-dominating energy.

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