THE INNER SITE-PERIMETER OF BARGRAPHS

AUBREY BLECHER, CHARLOTTE BRENNAN[1], AND ARNOLD KNOPFMACHER[1]

Abstract. Bargraphs are column convex polyominoes, where the lower edge lies on a horizontal axis. We consider the inner site-perimeter, which is the total number of cells inside the bargraph that have at least one edge in common with an outside cell and obtain the generating function that counts this statistic. From this we find the average inner perimeter and the asymptotic expression for this average as the semi-perimeter tends to infinity. We finally consider those bargraphs where the inner site-perimeter is exactly equal to the area of the bargraph.

1. Introduction

Bargraphs are column convex polyominoes, where the lower edge lies on a horizontal axis. They are drawn on a regular planar lattice grid and are made up of square cells. A bargraph is uniquely defined by the height or the number of cells in each column. The area of the bargraph is the total number of cells it contains while its perimeter is the number of edges that simultaneously are incident on a cell inside the bargraph and a cell outside. The enumeration of polyominoes according to their area and perimeter has been well documented and researched and is a basic problem in combinatorics [5]. The site-perimeter is the number of nearest-neighbour cells outside the boundary of the bargraph. The site-perimeter for directed animals, bargraphs and staircase polygons was investigated in [1, 4, 6]. We consider a related parameter, the *inner site-perimeter*. It is defined to be the number of cells inside the bargraph that have at least one edge in common with an outside cell. In the case of integer compositions, which are equivalent to bargraphs counted by area instead of semi-perimeter, the inner site-perimeter has recently been studied in [3, 11]. We shall rename the site-perimeter described above as the outer site-perimeter so as not to confuse it with the inner site-perimeter studied in this paper. We illustrate both site-perimeters, the outer and the inner, in Figure 1.

The outer site-perimeter is the sum of the cells marked by"o", whereas the inner site-perimeter is the sum of the cells marked by "x". The outer site-perimeter is 25 and the

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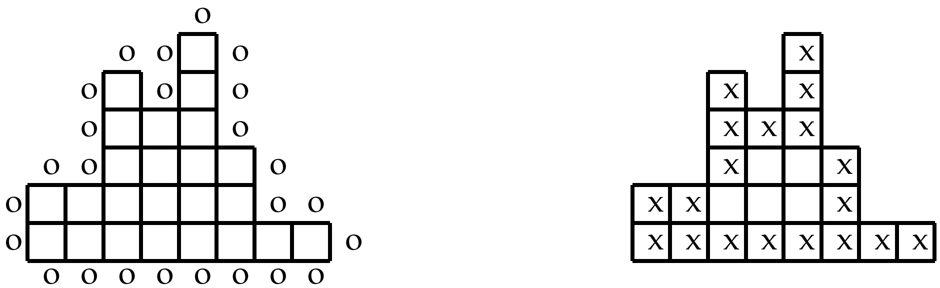

FIGURE 1. Outer and inner site-perimeters

inner site-perimeter is 19. The (ordinary) perimeter is the number of edges making up the border of the bargraph. The perimeter is 30, which can be split into the horizontal perimeter of 16 and the vertical perimeter of 14.

2. Inner site-perimeter generating function

Let $B := B(x, y, p)$ be the generating function of bargraphs that enumerate the horizontal and vertical semi perimeters by x and y respectively and the inner site-perimeter by p. Thus, for example the bargraph in Figure 1 is represented by $x^8y^7p^{19}$. To obtain this generating function, we shall use the wasp-waist decomposition, which was used for example in [4] and by the present authors in [2]. The wasp-waist decomposition takes a general bargraph and split it into smaller bargraphs according to the first occurence of the first singleton block (i.e., a horizontal step at height one). It is illustrated in Figure 2 below.

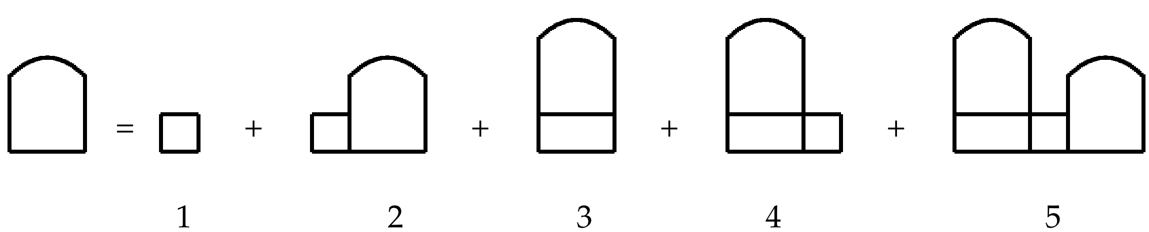

FIGURE 2. Wasp-waist decomposition of bargraphs

The numbers 1 to 5 below each drawing represent the case numbers. We shall refer to these cases throughout the paper. Let $B^R := B^R(x, y, p)$ be the generating function for the raised bargraphs of case 3. It represents the set of bargraphs that have a minimum height of 2.

The generating functions for cases 1, 2, 4 and 5 are clearly xyp, xpB, B^Rxp, B^RxpB/y respectively. Thus

$$B = xyp + xpB + B^R + B^Rxp + B^RxpB/y. \tag{2.1}$$

We shall find the generating function for case 3 in the next section.

3. Generating function for the raised case 3

For the more difficult case 3, we need to track the effect on the parameters of raising a bargraph. We distinguish two special situations, namely where we raise only a single row or where we raise a bargraph of height at least 2, (i.e., that with generating function B^R). This will form the basis for the method we use, namely expressing any bargraph as an alternating sequence of singletons (rows of cells of height 1) and bargraphs of height at least 2. To ease the notation we shall use $\mathcal{S}$ to indicate a singleton and $\mathcal{B}^{\mathcal{R}}$ to indicate a raised bargraph, i.e., a bargraph of height at least 2. We shall decompose the bargraphs of case 3 into the following *raised* options while using the usual notation $^+$ and * for a non-empty sequence and a possibly empty sequence, respectively.

3.1. The component $\mathcal{S}^+\mathcal{B}^{\mathcal{R}}$.

We consider a sequence of singletons followed by a bargraph of height at least two. This is expressed symbolically as $\mathcal{S}^+\mathcal{B}^{\mathcal{R}}$. We then raise the whole section up by one unit. This is illustrated below:

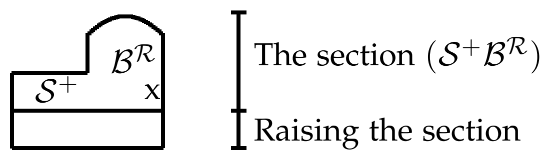

Figure 3. Diagram for raised $\mathcal{S}^+\mathcal{B}^{\mathcal{R}}$

The double row of singletons yields the term $\frac{xp^2}{1-xp^2}$ since each column has a contribution of 2 cells to the inner site-perimeter. After raising $\mathcal{B}^{\mathcal{R}}$, there is an extra cell in the inner site-perimeter marked by x in the diagram and it is counted by p in the generating function. Thus the generating function for such bargraphs is

$$C := \frac{xyp^3B^R}{1-xp^2}. \tag{3.1}$$

3.2. The sequence $(\mathcal{S}^+\mathcal{B}^{\mathcal{R}})^+$.

We now consider a sequence of components from the previous subsection, described symbolically as $(\mathcal{S}^+\mathcal{B}^{\mathcal{R}})^+$ and then raise that sequence. The diagram below illustrates this case:

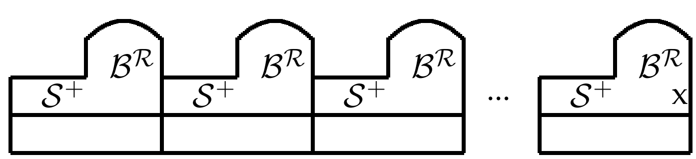

Figure 4. Diagram for raised $(\mathcal{S}^+\mathcal{B}^{\mathcal{R}})^+$

So C in (3.1) is the generating function for the rightmost component of the sequence; for the other members of the sequence the generating function is

$$A := \frac{xp^2 B^R}{1 - xp^2}, \tag{3.2}$$

as these components do not have the extra cell in the inner site-perimeter, that was marked by “x" in Figure 3.

Thus $C = pyA$. Therefore the generating function for the whole raised sequence $(\mathcal{S}^+\mathcal{B}^\mathcal{R})^+$ is

$$\frac{C}{1 - A/y}. \tag{3.3}$$

We need to divide A by y since the whole sequence was raised via the component with generating function C and therefore does not need to be raised again within the generating function A.

3.3. **The sequence $\mathcal{B}^\mathcal{R}(\mathcal{S}^+\mathcal{B}^\mathcal{R})^*$.** We consider a possibly empty sequence $(\mathcal{S}^+\mathcal{B}^\mathcal{R})^*$ which we prepend with a bargraph of height at least 2; hence we are considering $\mathcal{B}^\mathcal{R}(\mathcal{S}^+\mathcal{B}^\mathcal{R})^*$ and finally we raise the whole sequence.

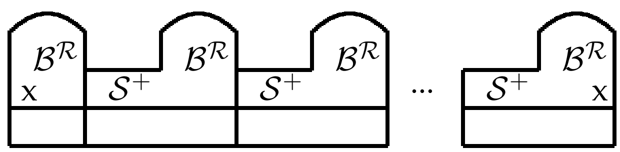

FIGURE 5. Diagram for raised $\mathcal{B}^\mathcal{R}(\mathcal{S}^+\mathcal{B}^\mathcal{R})^*$

We need to consider two subcases where $(\mathcal{S}^+\mathcal{B}^\mathcal{R})^*$ is empty.

Firstly, we consider the case where the prepending bargraph has only one column, its generating function after raising is

$$py\frac{y^2xp^2}{1 - yp}. \tag{3.4}$$

Next, we look at the case where the prepending bargraph consists of at least 2 columns. We need to subtract the case of bargraphs with only one column which after raising yields

$$p^2y\left(B^R - \frac{y^2xp^2}{1 - yp}\right). \tag{3.5}$$

Finally for the raised case where the sequence $(\mathcal{S}^+\mathcal{B}^\mathcal{R})^*$ is not empty the generating function is

$$pB^R\frac{C/y}{1 - A/y}. \tag{3.6}$$

Thus putting these 3 subcases together, the generating function for bargraphs of the form $\mathcal{B}^{\mathcal{R}}(\mathcal{S}^{+}\mathcal{B}^{\mathcal{R}})^{*}$ that are raised is

$$p^2y\left(B^R-\frac{y^2xp^2}{1-yp}\right)+py\frac{y^2xp^2}{1-yp}+pB^R\frac{C/y}{1-A/y}. \tag{3.7}$$

3.4. **The sequence** $(\mathcal{S}^{+}\mathcal{B}^{\mathcal{R}})^{*}\mathcal{S}^{+}$**.** Here, we consider a possibly empty sequence $(\mathcal{S}^{+}\mathcal{B}^{\mathcal{R}})^{*}$ as seen in Subsection 3.2, but we append a row of singletons on the right, hence we are considering $(\mathcal{S}^{+}\mathcal{B}^{\mathcal{R}})^{*}\mathcal{S}^{+}$ and thereafter we raise the whole sequence.

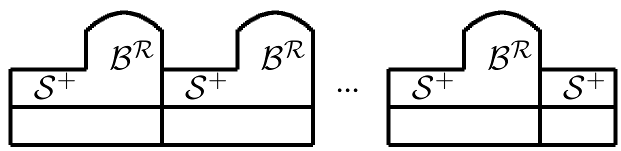

FIGURE 6. Diagram for raised $(\mathcal{S}^{+}\mathcal{B}^{\mathcal{R}})^{*}\mathcal{S}^{+}$

We first consider the subcase where $(S^{+}\,B^{R})^{*}$ is empty; thus we only have a double row of singletons with at least one column with semi-perimeter 2. Therefore the generating function is

$$\frac{y^2xp^2}{1-xp^2}. \tag{3.8}$$

Now for the case where $(\mathcal{S}^{+}\mathcal{B}^{\mathcal{R}})^{*}$ is not empty. The generating functions for this non-empty sequence and for the rows of singletons are $\frac{Ay}{1-A/y}$ and $\frac{xp^2}{1-xp^2}$ respectively. Putting these two cases together, we have

$$\frac{y^2xp^2}{1-xp^2}+\frac{Ayxp^2}{(1-A/y)(1-xp^2)}. \tag{3.9}$$

3.5. **The sequence** $\mathcal{B}^{\mathcal{R}}(\mathcal{S}^{+}\mathcal{B}^{\mathcal{R}})^{*}\mathcal{S}^{+}$**.** Finally, we consider the sequence $\mathcal{B}^{\mathcal{R}}(\mathcal{S}^{+}\mathcal{B}^{\mathcal{R}})^{*}$ described in Subsection 3.3, and append a row of singletons on the right, yielding $\mathcal{B}^{\mathcal{R}}(\mathcal{S}^{+}\mathcal{B}^{\mathcal{R}})^{*}\mathcal{S}^{+}$, which we then raise. This is illustrated below

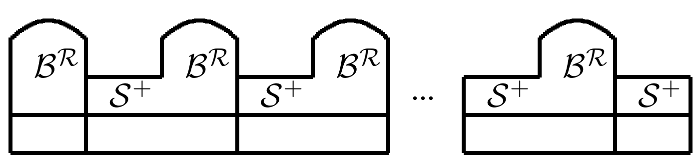

FIGURE 7. Diagram for raised $\mathcal{B}^{\mathcal{R}}(\mathcal{S}^{+}\mathcal{B}^{\mathcal{R}})^{*}\mathcal{S}^{+}$

It can also be represented by the non-empty sequence $(\mathcal{B}^{\mathcal{R}}\mathcal{S}^{+})^{+}$. Thus the generating function for the raised sequence $\mathcal{B}^{\mathcal{R}}(\mathcal{S}^{+}\mathcal{B}^{\mathcal{R}})^{*}\mathcal{S}^{+}$ is by symmetry the same as for the raised sequence $(\mathcal{S}^+\mathcal{B}^\mathcal{R})^+$. From (3.3) we have the following generating function

$$\frac{C}{1-A/y}. \tag{3.10}$$

3.6. **The full generating function for case 3.** We have considered all options and are now in a position to evaluate the full generating function B^R for $\mathcal{B}^\mathcal{R}$. Adding the generating functions (3.3), (3.7), (3.9) and (3.10) we obtain

$$\begin{aligned} B^R = \frac{2C}{1-A/y} + p^2y\left(B^R - \frac{y^2xp^2}{1-yp}\right) + py\frac{y^2xp^2}{1-yp} + pB^R\frac{C/y}{1-A/y} + \\ + \frac{y^2xp^2}{1-xp^2} + \frac{Ayxp^2}{(1-A/y)(1-xp^2)}. \end{aligned} \tag{3.11}$$

Solving for B^R we obtain

$$B^R = \frac{1}{2p^2x(1-py)}(M - y\sqrt{N}), \tag{3.12}$$

where

$$M = y - p^2xy - py^2 - p^2y^2 - p^3xy^2 + p^4xy^2 + p^3y^3 + 2p^4xy^3 - p^5xy^3 + p^5x^2y^3 - p^6x^2y^3$$

and

$$\begin{aligned} N &= 4p^4x^2y(-1+py)\left(1 - p^2y - p^3xy + p^4xy\right) \\ &\quad + (1 - py - p^3(x-y)y - p^5(1-x)xy^2 - p^6x^2y^2 - p^2(x+y) + p^4xy(1+2y))^2. \end{aligned}$$

4. Generating function for the inner site-perimeter

Adding the five cases of the wasp-waist decomposition (illustrated in Figure 2), we have as shown in (2.1)

$$B = xyp + xpB + B^R + B^Rxp + B^RxpB/y.$$

Substituting the expression for B^R and then solving for B we obtain the following result:

Theorem 1. *The generating function $B(x,y,p)$ for the inner site-perimeter of bargraphs, where the horizontal and vertical semi-perimeters are counted by x and y respectively and the inner site-perimeter is counted by p, is given by*

$$B(x,y,p) = \frac{Q}{R}$$

where

$$\begin{aligned} Q =& y(-y - pxy + py^2 + p^2xy + p^2y^2 + p^2xy^2 - p^3y^3 - p^3x^2y + 2p^3xy^2 - p^4xy^2 + 3p^4x^2y^2 \\ & - 3p^4xy^3 - p^5x^2y^2 + p^5xy^3 - 3p^5x^2y^3 + 2p^6x^2y^3 - p^6x^3y^3 + p^7x^3y^3 + (1+px)y\sqrt{X(p)}) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} R =& px\Big(y - 2py - py^2 + p^2xy + p^2y^2 + p^3y^3 - 3p^3xy^2 + p^4xy^2 + 2p^4xy^3 \\ & - p^5xy^3 + p^5x^2y^3 - p^6x^2y^3 - y\sqrt{X(p)}\Big). \end{aligned} \tag{4.2}$$

and where

$$\begin{aligned} X(p) =& 4p^4x^2y(-1+py)(1-p^2y-p^3xy+p^4xy) \\ & + \Big(-1+py+p^2(x+y)+p^3(x-y)y-p^4xy(1+2y)-p^5(-1+x)xy^2+p^6x^2y^2\Big)^2. \end{aligned} \tag{4.3}$$

4.1. **Bargraphs counted by the inner site-perimeter only.** We substitute x and y by 1 in Theorem 1. This gives the generating function for bargraphs with a fixed p value.

$$\begin{aligned} & B(p) \\ & = \frac{(1-p)(1+p+p^2)(-1-p+2p^2+p^3) + (1+p)\sqrt{(1+p+p^2)(1+p-p^3)(1-2p-2p^2+4p^3-p^4-p^5)}}{p\left[(1-2p-2p^3+p^4+p^5) - \sqrt{(1+p+p^2)(1+p-p^3)(1-2p-2p^2+4p^3-p^4-p^5)}\right]}. \end{aligned} \tag{4.4}$$

The series expansion for the above expression is

$$p+2p^2+4p^3+8p^4+16p^5+33p^6+68p^7+142p^8+298p^9+629p^{10}+1333p^{11}+2836p^{12}+\cdots.$$

For example, there are 8 bargraphs with inner site-perimeter 4. We illustrate these 8 bargraphs below:

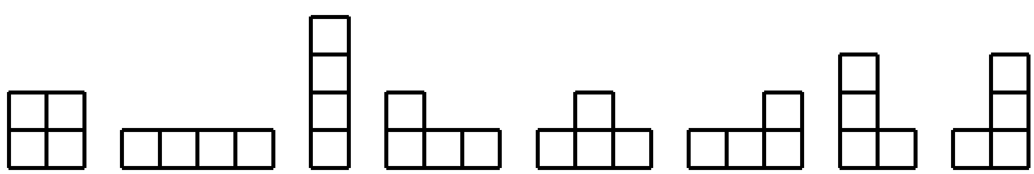

FIGURE 8. Bargraphs with inner site-perimeter 4

4.2. **Asymptotic for the number of bargraphs with inner site-perimeter n, as $n \to \infty$.** The dominant singularity of $B(p)$ is δ, the positive root of $1-2p-2p^2+4p^3-p^4-p^5$, numerically $\delta = 0.450021$, so that

$$B(p) \sim -\frac{2\sqrt{1+2\delta+2\delta^2-\delta^4-\delta^5}\,\sqrt{1-2p-2p^2+4p^3-p^4-p^5}}{(1-2\delta-2\delta^3+\delta^4+\delta^5)^2}.$$

By singularity analysis,

$$[p^n]B(p) \sim \frac{\sqrt{\delta(2+4\delta-12\delta^2+4\delta^3+5\delta^4)(1+2\delta+2\delta^2-\delta^4-\delta^5)}}{\sqrt{\pi}n^{3/2}(1-2\delta-2\delta^3+\delta^4+\delta^5)^2}\,\delta^{-n}, \quad \text{as } n \to \infty.$$

5. Asymptotic expression for the mean inner site-perimeter in the isotropic case

In this section we consider bargraphs with fixed semi-perimeter n and determine the average inner site-perimeter of all such bargraphs as $n \to \infty$. For this, we substitute y by x in $B(x,y,p)$ and find the isotropic generating function $B(x,x,p)$ with semi-perimeter tracked by x. We obtain

(5.1)
$$B(x,x,p) = \frac{-x+2p^2x^2+p^2x^3-p^4x^3-3p^5x^5+2p^6x^5-p^6x^6+p^7x^6+(1+px)x\sqrt{\tilde{X}}}{p\left(x-2px-px^2+2p^2x^2-2p^3x^3+p^4x^3+2p^4x^4-p^5x^4+p^5x^5-p^6x^5-x\sqrt{\tilde{X}}\right)},$$

where

$$\begin{aligned}\tilde{X} =& 4p^4x^3(-1+px)(1-p^2x-p^3x^2+p^4x^2) \\ &+\left(-1+px+2xp^2-p^4x^2(1+2x)+p^5(1-x)x^3+p^6x^4\right)^2.\end{aligned}$$

Here, we find the generating function for the total inner site-perimeter of bargraphs. For this we use the standard formula $\left.\dfrac{\partial B(x,x,p)}{\partial p}\right|_{p=1}$.

(5.2)
$$\left.\frac{\partial B(x,x,p)}{\partial p}\right|_{p=1} = 2\frac{1+2x-2x^2+2x^3+5x^4-5x^5+x^6-x^7+x^8+(1-x)(1+x-2x^2+x^5)\sqrt{Y}}{\sqrt{Y}\left(1-x+x^2-x^3+(1-x)\sqrt{Y}\right)^2}$$

where

$$Y = 1-4x+2x^2+x^4.$$

The series expansion for $\left.\frac{\partial B(x,x,p)}{\partial p}\right|_{p=1}$ is

$$x^2+4x^3+16x^4+60x^5+214x^6+741x^7+2523x^8+8508x^9+28529x^{10}+95343x^{11}+\cdots.$$

Let

$$\rho = \frac{1}{3}\left(-1-\frac{4\times 2^{2/3}}{(13+3\sqrt{33})^{1/3}}+\left(2(13+3\sqrt{33})\right)^{1/3}\right) = 0.295598\cdots. \tag{5.3}$$

be the positive root of $Y=0$. Then we find that

$$\left.\frac{\partial B(x,x,p)}{\partial p}\right|_{p=1} \sim \frac{(1-\rho-3\rho^2-\rho^3+4\rho^4-\rho^5-\rho^7)}{(1-\rho)(1+\rho^2)^2\sqrt{\rho(1-\rho-\rho^3)}}\left(1-\frac{x}{\rho}\right)^{-1/2}.$$

Extracting the coefficients of x^n gives

$$[x^n]\frac{\partial B(x,x,p)}{\partial p}\bigg|_{p=1} \sim \frac{1-\rho-3\rho^2-\rho^3+4\rho^4-\rho^5-\rho^7}{\sqrt{n\pi\rho(1-\rho-\rho^3)}(1-\rho)(1+\rho^2)^2}\,\rho^{-n}. \tag{5.4}$$

In order to obtain the average inner site-perimeter, we need to divide by the asymptotic number of bargraphs of semi perimeter n. We consider the generating function $B(x,y)$ that counts all bargraphs, it was explored in [2, 4] and found to be

$$B := B(x,y) = \frac{1-x-y-xy-\sqrt{(1-x-y-xy)^2-4x^2y}}{2x}. \tag{5.5}$$

In [2, 4] the authors found that

$$B(x,x) \sim \frac{-\sqrt{\rho(1-\rho-\rho^3)}}{\rho}\left(1-\frac{x}{\rho}\right)^{1/2} \text{ as } x \to \rho.$$

So that

$$[x^n]B(x,x) \sim \frac{\sqrt{\rho(1-\rho-\rho^3)}}{2\rho\sqrt{\pi}n^{3/2}}\rho^{-n}. \tag{5.6}$$

Thus after dividing (5.4) by (5.6), we get

Theorem 2. *For bargraphs of semi perimeter n the average inner site-perimeter is asymptotic to*

$$\frac{2(1-\rho-3\rho^2-\rho^3+4\rho^4-\rho^5-\rho^7)}{(1-\rho)(1+\rho^2)^2(1-\rho-\rho^3)}\,n = (1.57307\cdots)\,n$$

as $n \to \infty$.

6. Bargraphs where the inner site-perimeter equals the area

We finally consider bargraphs where the inner site-perimeter is exactly equal to the area of the bargraph. There are no cells without a "x" as described in Figure 1. If we call the group of cells inside the bargraph that are not part of the inner site-perimeter a "hole", then we want to look at bargraphs that have no "hole", (see Figure 9).

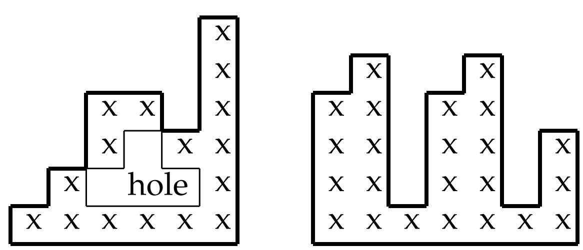

Figure 9. Bargraphs with and without a hole

Let $B_N := B_{No\,hole}(x,y)$ be the generating function for such bargraphs.

Using the wasp-waist decomposition from Figure 2, we clearly have

$$B_N = xy + xB_N + B_N^R + xB_N^R + \frac{x}{y} B_N^R B_N, \tag{6.1}$$

where $B_N^R := B_N^R(x, y)$ is the generating function for raised bargraphs with no holes (from case 3 of the wasp-waist decomposition). We shall compute B_N^R in the following subsection.

6.1. **Raised bargraphs with no holes.** We count these raised bargraphs with no hole according to their number of columns.

- One column: Clearly a bargraph with only one column does not have a hole. The generating function for that one column is
$$x \frac{y^2}{1-y}.$$
- Two columns: Let the height of the first column be i where $i \geq 2$, then for each i there are i choices for the height of the second column if the height of the first column is less than or equal to the height of the second column. We need to multiply by 2 to account for symmetry but we need to remove the case where the two columns have the same height as this case has been counted twice. Thus the generating function is
$$x^2 \left(2 \sum_{i=2}^{\infty} y^i \, (i-1) - \frac{y^2}{1-y} \right).$$
- At least 3 columns: In order to avoid a hole, we cannot have 3 adjacent columns of height a, b and c where $a, c \geq 2$ and $b \geq 3$. Thus for a *raised* bargraph with at least 3 columns not to have a hole, this is equivalent to one in which only the end columns may be of height at least 2 and all internal columns are of height 2. Examples of two such raised bargraphs are

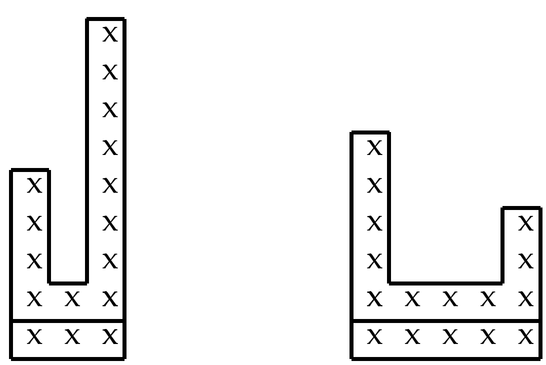

FIGURE 10. Raised bargraphs with at least 3 columns and no holes

The generating function for this case is therefore

$$x^3 \frac{1}{1-x} \frac{y^2}{(1-y)^2}.$$

Adding these 3 generating functions we have

$$B_N^R = \frac{xy^2}{1-y} + x^2 \left(\frac{2y^2}{(1-y)^2} - \frac{y^2}{1-y} \right) + \frac{x^3 y^2}{(1-x)(1-y)^2}.$$

We substitute the above expression for B_N^R into (6.1) and solve it to obtain

Theorem 3. *The generating function for bargraphs where the inner site-perimeter equals the area of the bargraph is*

$$B_N(x,y) = \frac{xy(-1+x+y-3xy-x^2y^2+x^3y^2)}{(1-2x)(1-y)^2-2x^3y^2+x^4y^2+x^2(1-3y+2y^2)}. \tag{6.2}$$

The series expansion for B_N begins with

$$x(y+y^2+y^3+y^4)+x^2(y+3y^2+5y^3+7y^4)+x^3(y+6y^2+12y^3+18y^4)+x^4(y+10y^2+24y^3+42y^4).$$

6.2. **Asymptotic expression.** Replacing y by x in (6.2) yields the isotropic generating function, $B_N(x,x)$

$$B_N(x,x) = \frac{x^2-2x^3+3x^4+x^6-x^7}{1-4x+6x^2-5x^3+2x^4-2x^5+x^6},$$

with series expansion that starts

$$x^2+2x^3+5x^4+13x^5+31x^6+\ldots.$$

The dominant root of the denominator of $B_N(x,x)$ is $\tau = 0.460626074\cdots$. Expanding $B_N(x,x)$ around τ we have

$$B_N(x,x) \sim \frac{\tau^2-2\tau^3+3\tau^4+\tau^6-\tau^7}{(x-\tau)(-4+12\tau-15\tau^2+8\tau^3-10\tau^4+6\tau^5)}.$$

Thus by singularity analysis

$$[x^n]B_N(x,x) \sim \frac{\tau^2-2\tau^3+3\tau^4+\tau^6-\tau^7}{(4-12\tau+15\tau^2-8\tau^3+10\tau^4-6\tau^5)}\,\tau^{-n-1}.$$

Dividing by the asymptotic total number of bargraphs, as seen in (5.6), the proportion of bargraphs with no holes is approximately

$$0.664736 \times 0.64173^n\, n^{3/2}.$$

We can therefore conclude

Theorem 4. *The proportion of bargraphs of semi-perimeter n with no holes tends to zero at an exponential rate as n tends to infinity.*

References

[1] A. Bacher. Average site perimeter of directed animals on the two-dimensional lattices, *Discrete Mathematics*, **312**, (2012), 1038–1058.

[2] A. Blecher, C. Brennan and A. Knopfmacher. Combinatorial parameters in bargraphs, *Quaestiones Mathematicae*, **39:5**, (2016), 619–635.

[3] A. Blecher, C. Brennan and A. Knopfmacher. The inner site-perimeter of compositions, *Quaestiones Mathematicae*, in press.

[4] M. Bousquet-Mélou and A. Rechnitzer, The site-perimeter of bargraphs, *Advances in Applied Mathematics*, **31**, (2003), 86–112.

[5] M. Bousquet-Mélou, R. Brak. Exactly solved models of polyominoes and polygons, Chapter 3 of Polygons, Polyominoes and Polycubes, *Lecture notes in physics*, volume 775, 43–78, Springer, Berlin, Heidelberg 2009.

[6] M. Delest, D. Gouyou-Beauchamps and B. Vauquelin, Enumeration of parallelogram polyominoes with given bond and site perimeter, *Graphs and Combinatorics*, **3**, (1987), 325–339.

[7] P. Duchon. q-grammars and wall polyominoes, *Ann. Comb.*, **3**, (1999), 311–321.

[8] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, 2009.

[9] A. Geraschenko, An investigation of skyline polynomials. http://people.brandeis.edu/ gessel/47a/geraschenko.pdf

[10] E. J. Janse van Rensburg and P. Rechnitzer. Exchange symmetries in Motzkin path and bargraph models of copolymer adsorption. *Electronic Journal of Combinatorics*, **9**, (2002), R20.

[11] T. Mansour. Border and tangent cells in bargraphs, *Discrete Math. Letters*, **1**, (2019), 26–29.

[12] J. Osborn, T. Prellberg, Forcing adsorption of a tethered polymer by pulling, *Journal of Statistical Mechanics: Theory and experiment*, (2010), P09018.

[13] T. Prellberg, R. Brak. Critical exponents from nonlinear functional equations for partially directed cluster models. *J. Stat. Phys.*, **78**, (1995), 701–730.

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