PROOF OF SOME COMBINATORIAL IDENTITIES BY AN ANALYTIC METHOD

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Abstract. We prove some combinatorial identities by an analytic method. We use the property that singular integrals of particular functions include binomial coefficients. In this paper, we prove combinatorial identities from the fact that two results of the particular function calculated as two ways using the residue theorem in the complex function theory are the same. These combinatorial identities are the generalization of a combinatorial identity that has been already obtained

1. Introduction

Gould [5] gave the following combinatorial identities ((3.63) and (3.117) in [5])

(1) $$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k\binom{x}{k}\binom{2x-2k}{n-2k}=\binom{x}{n}2^n,$$

(2) $$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k\binom{n}{k}\binom{2n-2k}{n}=2^n,$$

and, then Sprugnoli [14] proved (1) and (2) using Riodan arrays. (2) follows from (1) for $x = n$. And, as Riordan [13] showed that two-term sequence

(3) $$f_{mp}=\sum_{k=0}^{K}(-1)^k\binom{m}{k}\binom{2m-2k}{p-2k}, K=min\left\{m,\left\lfloor\frac{p}{2}\right\rfloor\right\}$$

satisfies the recurrence relation

$$f_{m,p}=f_{m-1,p}+2f_{m-1,p-1}$$

and the solution of this recurrence relation is $f_{m,p} = 2^p\binom{m}{p}$, he proved the following combinatorial identity equals to (1)

$$\sum_{k=0}^{K}(-1)^k\binom{m}{k}\binom{2m-2k}{p-2k}=2^p\binom{m}{p}, K=min\left\{m,\left\lfloor\frac{p}{2}\right\rfloor\right\}$$

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and, setting $m = p$ in above identity, the following combinatorial identity equals to (2) is derived.

$$\sum_{k=0}^{P}(-1)^k\binom{p-k}{k}\binom{2p-2k}{p-k} = 2^p, P = \left\lfloor\frac{p}{2}\right\rfloor.$$

The main proof methods used widely in the derivation of new combinatorial identities are the combinatorial proof, Riordan array proof, generating function proof, bijective proof, and so on [2, 3, 4, 6, 7, 8, 10, 11, 12, 15].

In this paper, by using the analytic method, that is, the calculation of singular integrals by the residue theorem in the complex function theory, we derive following two combinatorial identities

$$\binom{2m+2}{m+1}\sum_{l=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^l\binom{m+1}{2l+1}\binom{m+1}{l+k+1}\binom{2m+2}{2l+2k+2}^{-1} = (-1)^k 2^{m+1}, \tag{4}$$

$$\binom{2p}{p}\sum_{l=0}^{\lfloor\frac{p}{2}\rfloor}(-1)^l\binom{p}{2l}\binom{p}{k+l}\binom{2p}{2l+2k}^{-1} = (-1)^k 2^p. \tag{5}$$

Setting $k = 0$ in (4) and (5), we obtain (2)

2. Some combinatorial identities

First, we denote some functions and symbols as follows.

$$\begin{aligned}
f(x) &:= (x^2+1)^{-1},\\
g_{m,\xi}(x) &:= \frac{(-1)^{m+1}m!}{2i}\left[\frac{1}{(x-\xi+i)^{m+1}} - \frac{1}{(x-\xi-i)^{m+1}}\right],\\
h_{p,\xi}(x) &:= \frac{1}{2}\left[\frac{1}{(x-\xi+i)^{p}} + \frac{1}{(x-\xi-i)^{p}}\right],\\
A_{m,\xi}^{(n)} &:= \int_{-\infty}^{+\infty} f(x)g_{m,\xi}(x)(x-\xi)^n dx,\\
B_{p,\xi}^{(n)} &:= \int_{-\infty}^{+\infty} f(x)h_{p,\xi}(x)(x-\xi)^n dx.
\end{aligned}$$

Lemma 2.1. *For any non-negative integers m, n (with $n \leq m+1$), the following relation holds.*

$$A_{m,\xi}^{(n)} = \frac{(-1)^{m+1}m!\pi}{2i}\left[\frac{(i-\xi)^n}{(2i-\xi)^{m+1}} - \frac{(-i-\xi)^n}{(-2i-\xi)^{m+1}}\right]. \tag{6}$$

Proof.

$$\begin{aligned} A_{m,\xi}^{(n)} &= \int_{-\infty}^{+\infty} f(x) g_{m,\xi}(x)(x-\xi)^n dx \\ &= \int_{-\infty}^{+\infty} (x^2+1)^{-1} \cdot \frac{(-1)^{m+1} m!}{2i} \left[\frac{1}{(x-\xi+i)^{m+1}} - \frac{1}{(x-\xi-i)^{m+1}}\right](x-\xi)^n dx, \end{aligned}$$

(7)

$$= \frac{(-1)^{m+1} m!}{2i} \left[\int_{-\infty}^{+\infty} (x^2+1)^{-1} \cdot \frac{(x-\xi)^n}{(x-\xi+i)^{m+1}} dx - \int_{-\infty}^{+\infty} (x^2+1)^{-1} \cdot \frac{(x-\xi)^n}{(x-\xi-i)^{m+1}} dx \right].$$

First of all, we will notice the first singular integral in the bracket of (7). For the complex function

$$R(z) = \frac{P(z)}{Q(z)} = (x^2+1)^{-1} \cdot \frac{(x-\xi)^n}{(x-\xi+i)^{m+1}},$$

the degree of the polynomial $Q(z) = (z^2+1)(z-\xi+i)^{m+1}$ in the denominator of $R(z)$ is $m+3$ and that of the polynomial $P(z) = (z-\xi)^n$ in the numerator is n. Furthermore, $(m+3)-n \geq 2$ by the assumptions on m and n, so by the Residue Theorem of Complex Analysis [1], the following equation holds.

$$\int_{-\infty}^{+\infty} (x^2+1)^{-1} \cdot \frac{(x-\xi)^n}{(x-\xi+i)^{m+1}} dx = 2\pi i Res\left((z^2+1)^{-1} \cdot \frac{(z-\xi)^n}{(z-\xi+i)^{m+1}}; i \right),$$

where $z = i$ represents the simple pole of $R(z)$ in the upper half-plane. Thus,

$$\begin{aligned} 2\pi i Res\left((z^2+1)^{-1} \cdot \frac{(z-\xi)^n}{(z-\xi+i)^{m+1}}; i \right) &= 2\pi i \cdot \lim_{z\to i} \left((z-i)(z^2+1)^{-1} \cdot \frac{(z-\xi)^n}{(z-\xi+i)^{m+1}} \right) \\ &= 2\pi i \cdot \lim_{z\to i} \left((z+i)^{-1} \cdot \frac{(z-\xi)^n}{(z-\xi+i)^{m+1}} \right) \\ &= 2\pi i \cdot \left((i+i)^{-1} \cdot \frac{(i-\xi)^n}{(i-\xi+i)^{m+1}} \right) \\ &= \pi \cdot \frac{(i-\xi)^n}{(2i-\xi)^{m+1}}. \end{aligned}$$

Next, we will estimate the second singular integral in the bracket of (7). Similar to the calculation of the first singular integral, $z = -i$ is the simple pole of the complex function

$$(z^2+1)^{-1}\cdot\frac{(z-\xi)^n}{(z-\xi-i)^{m+1}}$$

in the lower half-plane, so the following equations hold by the Residue Theorem.

$$\int_{-\infty}^{+\infty}(x^2+1)^{-1}\cdot\frac{(x-\xi)^n}{(x-\xi-i)^{m+1}}dx = 2\pi i Res\left((z^2+1)^{-1}\cdot\frac{(z-\xi)^n}{(z-\xi-i)^{m+1}};-i\right)$$

$$\begin{aligned}
&= (-2\pi i)\cdot\lim_{z\to -i}\left((z+i)(z^2+1)^{-1}\cdot\frac{(z-\xi)^n}{(z-\xi-i)^{m+1}}\right)\\
&= (-2\pi i)\cdot\lim_{z\to -i}\left((z-i)^{-1}\cdot\frac{(z-\xi)^n}{(z-\xi-i)^{m+1}}\right)\\
&= (-2\pi i)\cdot\left((-i-i)^{-1}\cdot\frac{(-i-\xi)^n}{(-i-\xi-i)^{m+1}}\right) = \pi\cdot\frac{(-i-\xi)^n}{(-2i-\xi)^{m+1}}.
\end{aligned}$$

Therefore, the lemma is true. □

Lemma 2.2. *For any non-negative integers $p, n (n \leq p)$, the following relation holds.*

$$B^{(n)}_{p,\xi} = \frac{\pi}{2}\left[\frac{(i-\xi)^n}{(2i-\xi)^p}+\frac{(-i-\xi)^n}{(-2i-\xi)^p}\right]. \tag{8}$$

Proof.

$$\begin{aligned}
B^{(n)}_{m,\xi} &= \int_{-\infty}^{+\infty} f(x)h_{p,\xi}(x)(x-\xi)^n dx\\
&= \int_{-\infty}^{+\infty}(x^2+1)^{-1}\cdot\frac{1}{2}\left[\frac{1}{(x-\xi+i)^p}+\frac{1}{(x-\xi-i)^p}\right](x-\xi)^n dx,
\end{aligned}$$

$$= \frac{1}{2}\left[\int_{-\infty}^{+\infty}(x^2+1)^{-1}\cdot\frac{(x-\xi)^n}{(x-\xi+i)^p}dx + \int_{-\infty}^{+\infty}(x^2+1)^{-1}\cdot\frac{(x-\xi)^n}{(x-\xi-i)^p}dx\right]. \tag{9}$$

First, we will compute the first singular integral in the bracket of (9). Let the complex function $R(z)$ be

$$R(z) = \frac{P(z)}{Q(z)} = (x^2+1)^{-1}\cdot\frac{(x-\xi)^n}{(x-\xi+i)^p}.$$

Similar to Lemma 2.1, the degree of the polynomial $Q(z) = (z^2+1)(z-\xi+i)^p$ in the denominator of $R(z)$ is $p+2$ and that of the polynomial $P(z) = (z-\xi)^n$ in the numerator is n. Moreover, $(p+2)-n \geq 2$ by the assumptions on p and n, and $z=i$ is the simple pole of $R(z)$ in the upper half-plane. So by the Residue Theorem, we have

$$\begin{aligned}\int_{-\infty}^{+\infty}(x^2+1)^{-1}\cdot\frac{(x-\xi)^n}{(x-\xi+i)^p}dx &= 2\pi i Res\left((z^2+1)^{-1}\cdot\frac{(z-\xi)^n}{(z-\xi+i)^p};i\right)\\ &= 2\pi i\cdot\lim_{z\to i}\left((z-i)(z^2+1)^{-1}\cdot\frac{(z-\xi)^n}{(z-\xi+i)^p}\right)\\ &= 2\pi i\cdot\lim_{z\to i}\left((z+i)^{-1}\cdot\frac{(z-\xi)^n}{(z-\xi+i)^p}\right)\\ &= 2\pi i\cdot\left((i+i)^{-1}\cdot\frac{(i-\xi)^n}{(i-\xi+i)^p}\right)=\pi\cdot\frac{(i-\xi)^n}{(2i-\xi)^p}\end{aligned}$$

.

And, $z=-i$ is the simple pole of the complex function

$$(z^2+1)^{-1}\cdot\frac{(z-\xi)^n}{(z-\xi-i)^p}$$

in the lower half-plane. So by the Residue Theorem, the second singular integral of (9) is computed as follows.

$$\begin{aligned}\int_{-\infty}^{+\infty}(x^2+1)^{-1}\cdot\frac{(x-\xi)^n}{(x-\xi-i)^p}dx &= 2\pi i Res\left((z^2+1)^{-1}\cdot\frac{(z-\xi)^n}{(z-\xi-i)^p};-i\right)\\ &= (-2\pi i)\cdot\lim_{z\to -i}\left((z+i)(z^2+1)^{-1}\cdot\frac{(z-\xi)^n}{(z-\xi-i)^p}\right)\\ &= (-2\pi i)\cdot\lim_{z\to i}\left((z-i)^{-1}\cdot\frac{(z-\xi)^n}{(z-\xi-i)^p}\right)\\ &= (-2\pi i)\cdot\left((-i-i)^{-1}\cdot\frac{(-i-\xi)^n}{(-i-\xi-i)^{m+1}}\right)=\pi\cdot\frac{(-i-\xi)^n}{(-2i-\xi)^p}.\end{aligned}$$

Therefore, the lemma is true. □

Theorem 2.3. *For any integer m and k (with $0 \leq k \leq \lfloor\frac{m}{2}\rfloor$) , the following combinatorial identity holds.*

$$\binom{2m+2}{m+1}\sum_{l=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^l\binom{m+1}{2l+1}\binom{m+1}{l+k+1}\binom{2m+2}{2l+2k+2}^{-1}=(-1)^k2^{m+1}.$$

Here, $\lfloor m\rfloor$ means the largest integer less than or equal to m .

Proof. For $\xi = 0$, by the binomial formula, $g_{m,\xi}(x)$ can be expanded as follows.

$$
\begin{aligned}
g_{m,0}(x) &= \frac{(-1)^{m+1}m!}{2i}\left[\frac{1}{(x+i)^{m+1}} - \frac{1}{(x-i)^{m+1}}\right] \\
&= \frac{(-1)^{m+1}m!}{2i}\cdot\frac{(x-i)^{m+1}-(x+i)^{m+1}}{(x^2+1)^{m+1}} \\
&= \frac{(-1)^{m+1}m!}{2i}\cdot\frac{\sum_{j=0}^{m+1}\binom{m+1}{j}x^{m+1-j}(-i)^j - \sum_{j=0}^{m+1}\binom{m+1}{j}x^{m+1-j}i^j}{(x^2+1)^{m+1}}
\end{aligned}
$$

$$
(10) \qquad = \frac{(-1)^{m+1}m!}{2i}\cdot\frac{\sum_{j=0}^{m+1}\binom{m+1}{j}x^{m+1-j}\left[(-i)^j - i^j\right]}{(x^2+1)^{m+1}}.
$$

Since $i^{2l} = (-i)^{2l}$, in the numerator of (10), the terms with j being even, that is, $j = 2l$, are omitted. On the other hand, $i^{2l+1} = (-1)^l \cdot i, (-i)^{2l+1} = (-1)^l \cdot (-i)$, so, in the numerator of (10), the terms with j being odd, that is, $j = 2l+1$, are as follows.

$$
\begin{aligned}
g_{m,0}(x) &= \frac{(-1)^{m+1}m!}{2i}\cdot\frac{2\sum_{l=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m+1}{2l+1}x^{m-2l}(-i)^{l+1}i}{(x^2+1)^{m+1}} \\
&= (-1)^{m+1}m!(x^2+1)^{-m-1}\sum_{l=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m+1}{2l+1}x^{m-2l}(-i)^{l+1} \\
&= (-1)^{m}m!(x^2+1)^{-m-1}\sum_{l=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^l\binom{m+1}{2l+1}x^{m-2l}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
A_{m,0}^{(m-2k)} &= \int_{-\infty}^{+\infty} f(x)g_{m,0}(x)x^{m-2k}dx \\
&= \int_{-\infty}^{+\infty}(x^2+1)^{-1}\left[(-1)^m m!(x^2+1)^{-m-1}\sum_{l=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^l\binom{m+1}{2l+1}x^{m-2l}\right]x^{m-2k}dx \\
&= \int_{-\infty}^{+\infty}\frac{(-1)^m m!}{(x^2+1)^{m+2}}\cdot\sum_{l=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^l\binom{m+1}{2l+1}x^{2m-2l-2k}dx \\
&= \sum_{l=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^{m+l}m!\binom{m+1}{2l+1}\int_{-\infty}^{+\infty}\frac{x^{2m-2l-2k}}{(x^2+1)^{m+2}}dx.
\end{aligned}
$$

Now we will calculate the singular integral

$$\int_{-\infty}^{+\infty} \frac{x^{2m-2l-2k}}{(x^2+1)^{m+2}} dx.$$

Using the notations

$$I_{\beta+1}(0) = 1, I_{\beta+1}(1) = \frac{1}{2\beta+1},$$

$$I_{\beta+1}(\alpha) = \frac{1 \cdot 3 \cdot \cdots \cdot (2\alpha - 1)}{(2\beta+1)(2\beta-1)\cdots(2\beta-2\alpha+3)} = \binom{\beta+1}{\alpha}\binom{2\beta+2}{2\alpha}^{-1},$$

for $\rho = 1, a = \infty$, (1) in [9, p. 575] implies

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{x^{2\alpha}}{(x^2+1)^{\beta+2}} dx &= \frac{1 \cdot 3 \cdot \cdots \cdot (2\alpha-1)}{(2\beta+1)(2\beta-1)\cdot \cdots \cdot(2\beta-2\alpha+3)} \int_{-\infty}^{+\infty} \frac{1}{(x^2+1)^{\beta+2}} dx \\ &= I_{\beta+1}(\alpha) \int_{-\infty}^{+\infty} \frac{1}{(x^2+1)^{\beta+2}} dx,\end{aligned}$$

and (2) in [9, p. 576] implies

$$\int_{-\infty}^{+\infty} \frac{1}{(x^2+1)^{\beta+2}} dx = \frac{\pi}{2^{2\beta+2}} \frac{(2\beta+2)!}{[(\beta+1)!]^2} = \frac{\pi}{2^{2\beta+2}} \binom{2\beta+2}{\beta+1},$$

where α is a positive integer, β is a nonnegative integer and $\alpha < \beta + 2$.

Therefore, the singular integral $\int_{-\infty}^{+\infty} \frac{x^{2m-2l-2k}}{(x^2+1)^{m+2}} dx$ is as follows.

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{x^{2m-2l-2k}}{(x^2+1)^{m+2}} dx &= \frac{\pi}{2^{2m+2}} \binom{2m+2}{m+1} \cdot I_{m+1}(m-l-k) \\ &= \frac{\pi}{2^{2m+2}} \binom{2m+2}{m+1} \binom{m+1}{l+k+1} \binom{2m+2}{2l+2k+2}^{-1}.\end{aligned}$$

Thus,

$$\begin{aligned}A_{m,0}^{(m-2k)} &= \sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^{m+l} m! \binom{m+1}{2l+1} \left[\frac{\pi}{2^{2m+2}} \binom{2m+2}{m+1} \cdot I_{m+1}(m-l-k) \right] \\ &= \pi \sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^{m+l} m! 2^{-2m-2} \binom{m+1}{2l+1} \binom{2m+2}{m+1} \binom{m+1}{l+k+1} \binom{2m+2}{2l+2k+2}^{-1}.\end{aligned} \tag{11}$$

And, by (6) of Lemma 2.1,

$$\begin{aligned} A_{m,0}^{(m-2k)} &= \frac{(-1)^{m+1}m!\pi}{2i}\left[\frac{i^{m-2k}}{(2i)^{m+1}}-\frac{(-i)^{m-2k}}{(-2i)^{m+1}}\right] \\ &= \frac{(-1)^{m+1}m!\pi}{2i}\frac{i^{m-2k}\cdot(-2i)^{m+1}-(-i)^{m-2k}\cdot(2i)^{m+1}}{(2i)^{m+1}\cdot(-2i)^{m+1}} \end{aligned}$$

$$= (-1)^{m+1}m!\pi\cdot\frac{(-1)^{m-k+1}\cdot 2^{m+2}}{(-1)^m\cdot 2^{2m+3}} = \pi(-1)^{m-k}\cdot m!2^{-m-1}. \tag{12}$$

Comparing (11) and (12),

$$\binom{2m+2}{m+1}\sum_{l=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^l\binom{m+1}{2l+1}\binom{m+1}{l+k+1}\binom{2m+2}{2l+2k+2}^{-1} = (-1)^k 2^{m+1}.$$

The theorem is true. □

Theorem 2.4. *For any non-negative integer p and k (with $0\le k\le\lfloor\frac{p}{2}\rfloor$), the following combinatorial identity holds.*

$$\binom{2p}{p}\sum_{l=0}^{\lfloor\frac{p}{2}\rfloor}(-1)^l\binom{p}{2l}\binom{p}{l+k}\binom{2p}{2l+2k}^{-1} = (-1)^k 2^p.$$

Proof. For $\xi = 0$, by the binomial formula, $h_{p,\xi}(x)$ can be expanded as follows.

$$\begin{aligned} h_{p,0}(x) &= \frac{1}{2}\left[\frac{1}{(x+i)^p}+\frac{1}{(x-i)^p}\right] \\ &= \frac{1}{2}\cdot\frac{(x-i)^p+(x+i)^p}{(x^2+1)^p} \\ &= \frac{1}{2(x^2+1)^p}\cdot\left[\sum_{j=0}^{p}\binom{p}{j}x^{p-j}(-i)^j+\sum_{j=0}^{p}\binom{p}{j}x^{p-j}i^j\right] \end{aligned}$$

$$= \frac{1}{2(x^2+1)^p}\cdot\sum_{j=0}^{p}\binom{p}{j}x^{p-j}\left[(-i)^j+i^j\right]. \tag{13}$$

Unlike Theorem 2.3, in (13), the terms with j being odd are omitted and those with j even are the only remaining, so

$$\begin{aligned} h_{m,0}(x) &= \frac{1}{2(x^2+1)^p}\cdot\sum_{l=0}^{\lfloor\frac{p}{2}\rfloor}2\cdot(-1)^l\binom{p}{2l}x^{p-2l} \\ &= \frac{1}{(x^2+1)^p}\cdot\sum_{l=0}^{\lfloor\frac{p}{2}\rfloor}\cdot(-1)^l\binom{p}{2l}x^{p-2l}. \end{aligned}$$

Thus,

$$\begin{aligned}
B_{p,0}^{(p-2k)} &= \int_{-\infty}^{+\infty} f(x)h_{p,0}(x)x^{p-2k}dx \\
&= \int_{-\infty}^{+\infty} (x^2+1)^{-1}\left[\frac{1}{(x^2+1)^p}\sum_{l=0}^{\lfloor \frac{p}{2}\rfloor}(-1)^l\binom{p}{2l}x^{p-2l}\right]x^{p-2k}dx \\
&= \int_{-\infty}^{+\infty} (x^2+1)^{-p-1}\cdot\left[\sum_{l=0}^{\lfloor \frac{m}{2}\rfloor}(-1)^l\binom{p}{2l}x^{2p-2l-2k}dx\right] \\
&= \sum_{l=0}^{\lfloor \frac{p}{2}\rfloor}(-1)^l\binom{p}{2l}\int_{-\infty}^{+\infty}(x^2+1)^{-p-1}x^{2p-2l-2k}dx.
\end{aligned}$$

Similarly to Theorem 2.3, we will calculate the singular integral

$$\int_{-\infty}^{+\infty}(x^2+1)^{-p-1}x^{2p-2l-2k}dx.$$

$$\int_{-\infty}^{+\infty}\frac{x^{2p-2l-2k}}{(x^2+1)^{p+1}}x^{2p-2l-2k}dx = \frac{\pi}{2^{2p}}\binom{2p}{p}\binom{p}{k+l}\binom{2p}{2k+2l}^{-1}.$$

Therefore, $B_{p,0}^{(p-2k)}$ is as follows.

$$\begin{aligned}
B_{p,0}^{(p-2k)} &= \sum_{l=0}^{\lfloor \frac{p}{2}\rfloor}(-1)^l\pi\binom{p}{2l}\binom{2p}{p}2^{-2p}I_{p(p-l-k)} \\
&= \binom{2p}{p}\sum_{l=0}^{\lfloor \frac{p}{2}\rfloor}(-1)^l\pi\binom{p}{2l}2^{-2p}\cdot\binom{p}{l+k}\binom{2p}{2l+2k}^{-1} \qquad (14)
\end{aligned}$$

On the other hand, by (8) of Lemma 2.2,

$$\begin{aligned}
B_{m,0}^{(p-2k)} &= \frac{\pi}{2}\left[\frac{i^{p-2k}}{(2i)^p}+\frac{(-i)^{p-2k}}{(-2i)^p}\right] \\
&= \frac{\pi}{2}\cdot\frac{i^{p-2k}\cdot(-2i)^p+(-i)^{p-2k}\cdot(2i)^p}{(2i)^p\cdot(-2i)^p} \\
&= \frac{\pi}{2}\cdot\frac{2\cdot i^{2p-2k}\cdot(-1)^p\cdot 2^p}{2^{2p}\cdot(-1)^p\cdot i^{2p}} = \pi(-1)^k\cdot 2^{-p}. \qquad (15)
\end{aligned}$$

Comparing (14) and (15),

$$\binom{2p}{p}\sum_{l=0}^{\lfloor\frac{p}{2}\rfloor}(-1)^l\pi\binom{p}{2l}2^{-2p}\cdot\binom{p}{l+k}\binom{2p}{2l+2k}^{-1} = \pi(-1)^k\cdot 2^{-p},$$

$$\binom{2p}{p}\sum_{l=0}^{\lfloor\frac{p}{2}\rfloor}(-1)^l\binom{p}{2l}\binom{p}{l+k}\binom{2p}{2l+2k}^{-1} = (-1)^k\cdot 2^{p}.$$

Hence, the theorem is proved. □

The combinatorial identity from the following Corollary 2.5 (2) is known already and is a special case of (4) and (5).

Corollary 2.5. *For any non-negative integers m, the following combinatorial identity holds.*

$$\sum_{l=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^l\binom{2m-2l}{m-l}\binom{m-l}{l}=2^m.$$

Proof. If we set $k=0$ in (4) , then the left-hand side is

$$\binom{2m+2}{m+1}\sum_{l=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^l\binom{m+1}{2l+1}\binom{m+1}{l+k+1}\binom{2m+2}{2l+2k+2}^{-1}$$

$$= \binom{2m+2}{m+1}\sum_{l=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^l\binom{m+1}{2l+1}\binom{m+1}{l+1}\binom{2m+2}{2l+2}^{-1}$$

$$= 2\sum_{l=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^l\binom{2m-2l}{m-l}\binom{m-l}{l}.$$

And, the right-hand side is 2^{m+1}, so the corollary is true. □

Setting $k=0$ in (5), we also obtain (2).

Corollary 2.6. *For any non-negative integer n, the following combinatorial identity holds.*

$$\sum_{l=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^l\binom{2n+1}{n-l}\binom{2n+2l+2}{2l+1}=(-1)^n\cdot 2^{2n+1}. \tag{16}$$

Proof. Setting $p=2n, k=n$ in (4), (16) follows immediately from (4). □

Corollary 2.7. *For any non-negative integer n the following combinatorial identity holds.*

$$\sum_{l=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^l\binom{2n+2l}{n+l}\binom{n+l}{n-l}=(-1)^n\cdot 2^{2n}. \tag{17}$$

Proof. Setting $p=2n, k=n$ in (5), (17) follows immediately from (5). □

3. Conclusion

In (4) and (5), by appropriately changing parameters k, m, and p, different combinatorial identities are obtained. Also, if we properly choose such functions as $f(x), g_{m,\xi}, h_{p,\xi}$, $A^{(n)}_{m,\xi}$, and $B^{(n)}_{p,\xi}$, we can obtain better combinatorial identities.

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