A NEW CLASS OF SYMMETRIC FUNCTIONS OF BINARY PRODUCTS OF TRIBONACCI NUMBERS AND OTHER WELL-KNOWN NUMBERS

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Abstract. In this paper, we will recover the generating functions of Tribonacci numbers and Chebychev polynomials of first and second kind. By making use of the operator defined in this paper, we give some new generating functions for the binary products of Tribonacci with some remarkable numbers and polynomials. The technique used here is based on the theory of the so-called symmetric functions.

1. Introduction

Recently, Fibonacci and Lucas numbers have been investigated very largely and authors tried to develop and give some directions to mathematical calculations using these type of special numbers [16, 22, 29, 32]. One of these directions goes through to the Tribonacci and the Tribonacci-Lucas numbers. In fact Tribonacci numbers have been firstly defined by M. Feinberg in 1963 and then some important properties for these numbers have been investigated by [18, 21, 25, 28, 33]. On the other hand, Tribonacci-Lucas numbers have been introduced and investigated by Elia in [17]. In addition, there exists another term, namely incomplete, on Fibonacci, Lucas and Tribonacci numbers. As a brief background, the incomplete Fibonacci, Lucas and Tribonacci numbers were introduced by authors [19, 26, 27], and further the generating functions of these numbers were presented.

Definition 1. For \( n \geq 2 \), it is known that while the Tribonacci sequence \( \{T_n\}_{n \in \mathbb{N}} \) is defined by

\[
\begin{cases}
T_n = T_{n-1} + T_{n-2} + T_{n-3}, n \geq 3 \\
T_0 = T_1 = 1, T_2 = 2
\end{cases}
\]

(1.1)

It is also well-known that each of the Tribonacci numbers is actually a linear combination of

\[
T_n = -\frac{(r_2 + r_3 - r_2 r_3 - 2)}{(r_1 - r_2)(r_1 - r_3)} r_1^n + \frac{(r_1 + r_3 - r_1 r_3 - 2)}{(r_1 - r_2)(r_2 - r_3)} r_2^n - \frac{(r_1 + r_2 - r_1 r_2 - 2)}{(r_1 - r_3)(r_2 - r_3)} r_3^n,
\]

(1.2)

where \( r_1, r_2, \) and \( r_3 \) are roots of the characteristic equations of (1.1) such that

\begin{footnotesize}
\footnotesize
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\end{footnotesize}
\[
\begin{align*}
  r_1 &= \frac{1}{3} \left[ 1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}} \right] = 1.8393, \\
  r_2 &= \frac{1}{3} \left[ 1 + w\sqrt[3]{19 + 3\sqrt{33}} + w^2\sqrt[3]{19 - 3\sqrt{33}} \right] = -0.41964 + 0.60629i, \\
  r_3 &= \frac{1}{3} \left[ 1 + w^2\sqrt[3]{19 + 3\sqrt{33}} + w\sqrt[3]{19 - 3\sqrt{33}} \right] = -0.41964 - 0.60629i,
\end{align*}
\]
with \( w = \frac{-1 + i\sqrt{3}}{2} \). Meanwhile we note that equations in (1.2) are called the Binet’s formulas for Tribonacci numbers.

**Definition 2.** Let \( k \) be a positive real number. Then, the recurrence relation of generalized \( k \)-Fibonacci numbers \( \{G_{k,n}\}_{n \in \mathbb{N}} \) is defined by

\[
G_{k,n+1} = kG_{k,n} + G_{k,n-1}, \quad n \geq 1
\]

with initial conditions \( G_{k,0} = a, \ G_{k,1} = b \ (a, b \in \mathbb{R}) \).

Generalized \( k \)-Fibonacci number is called to each element of Generalized \( k \)-Fibonacci sequence. Taking \( a = 1, b = k \) and \( a = 2, b = k \) gives \( k \)-Fibonacci numbers and \( k \)-Lucas numbers, respectively. A few special values for Generalized \( k \)-Fibonacci sequence \( \{G_{k,n}\}_{n \in \mathbb{N}} \) are listed below:

i. If \( k = 1 \), then we have generalized Fibonacci numbers \( \{G_{1,n}\} \)

\[
\{G_{1,n}\} = \{a, b, a + b, a + 2b, 2a + 3b, ...\}.
\]

- Putting \( a = 1, b = 1 \) reduces to Fibonacci numbers known as \( \{F_n\} = \{1, 1, 2, 3, 5, ...\} \).
- Substituting \( a = 2, b = 1 \) yields Lucas numbers given by \( \{L_n\} = \{2, 1, 3, 4, 7, 11, ...\} \).

ii. If \( k = 2 \), then we have generalized Pell numbers

\[
\{G_{2,n}\} = \{a, b, a + 2b, a + 5b, 2a + 12b, 12a + 29b, ...\}.
\]

- Taking \( a = 0, b = 1 \) gives Pell numbers given by \( \{P_n\} = \{0, 1, 2, 5, 12, 29, ...\} \).
- In the case when \( a = 2 \) and \( b = 2 \), it reduces to Pell-Lucas numbers known as \( \{Q_n\} = \{2, 2, 6, 14, 34, 82, ...\} \).

In this part, we define \( k \)-Pell and \( k \)-Pell Lucas numbers, Mersenne numbers and \( k \)-Jacobsthal-Lucas numbers.

**Definition 3.** [30] For \( n \in \mathbb{N} \), the \( k \)-Mersenne numbers, say \( \{M_{k,n}\}_{n \in \mathbb{N}} \) is defined recurrently by:

\[
\begin{cases}
  M_{k,n+1} = 3kM_{k,n} - 2M_{k,n-1} & \text{for all } \ n \geq 1, \\
  M_0 = 0, \ M_1 = 1.
\end{cases}
\]

**Definition 4.** [22] For \( n \in \mathbb{N} \), the \( k \)-Pell numbers, denoted by \( \{P_{k,n}\}_{n \in \mathbb{N}} \) defined recursively by

\[
\{P_{k,n}\} = 2P_{k,n-1} + kP_{k,n-2}, \quad n \geq 2, P_{k,0} = 0, \ P_{k,1} = k.
\]
Definition 5. [22] We define $k$-Pell-Lucas numbers $\{Q_{k,n}\}_{n \in \mathbb{N}}$ by the following recurrence relation as
\[
\cdot Q_{k,n} = 2Q_{k,n-1} + kQ_{k,n-2}, \quad n \geq 2, \quad Q_{k,0} = 2, \quad Q_{k,1} = 2.
\]
Definition 6. We define $k$-Jacobsthal numbers $\{J_{k,n}\}_{n \in \mathbb{N}}$ by the following recurrence relation as
\[
\cdot J_{k,n} = kJ_{k,n-1} + 2J_{k,n-2}, \quad n \geq 2, \quad J_{k,0} = 0, \quad J_{k,1} = k.
\]
Definition 7. For $n \in \mathbb{N}$, the $k$-Jacobsthal Lucas numbers, denoted by $\{j_{k,n}\}_{n \in \mathbb{N}}$ defined recursively by
\[
\cdot j_{k,n} = kj_{k,n-1} + 2j_{k,n-2}, \quad n \geq 2, \quad j_{k,0} = 2, \quad j_{k,1} = k.
\]

The further contents of this paper are as follows: Section 1 gives the preliminary, in Section 2, we introduce new symmetric functions and some of its properties. We also give some more useful definitions which are used in the following sections. In Section 3, we prove our main result which relates the symmetric function defined in the previous section with the symmetrizing operator. This main theorem unifies several previously known results about the generating functions.

2. Definitions and some properties

In order to render the work self-contained we give the necessary preliminaries tools; we recall some definitions and results.

Definition 8. [24] Let $k$ and $n$ be two positive integers and $\{a_1, a_2, ..., a_n\}$ are set of given variables the $k$-th elementary symmetric function $e_k(a_1, a_2, ..., a_n)$ is defined by
\[
e_k(a_1, a_2, ..., a_n) = \sum_{i_1 + i_2 + ... + i_n = k} a_1^{i_1} a_2^{i_2} ... a_n^{i_n} \quad (0 \leq k \leq n)
\]
with $i_1, i_2, ..., i_n = 0$ or 1.

Definition 9. [24] Let $k$ and $n$ be two positive integers and $A = \{a_1, a_2, ..., a_n\}$ are set of given variables the $k$-th complete homogeneous symmetric function $h_k(a_1, a_2, ..., a_n)$ is defined by
\[
h_k(a_1, a_2, ..., a_n) = \sum_{i_1 + i_2 + ... + i_n = k} a_1^{i_1} a_2^{i_2} ... a_n^{i_n} \quad (0 \leq k \leq n)
\]
with $i_1, i_2, ..., i_n \geq 0$.

Remark 1. Set $e_0(a_1, a_2, ..., a_n) = 1$ and $h_0(a_1, a_2, ..., a_n) = 1$, by usual convention. For $k < 0$, we set $e_k(a_1, a_2, ..., a_n) = 0$ and $h_k(a_1, a_2, ..., a_n) = 0$.

Definition 10. [11] Let $A$ and $P$ be any two alphabets. We define $S_n(A - P)$ by the following form
\[
(2.1) \quad \frac{\Pi_{p \in P}(1 - pt)}{\Pi_{a \in A}(1 - at)} = \sum_{n=0}^{\infty} S_n(A - P)t^n,
\]
with the condition $S_n(A - P) = 0$ for $n < 0$. 
Equation (2.1) can be rewritten in the following form

\[
S_n(A - P)t^n = \left( \sum_{n=0}^{\infty} S_n(A)t^n \right) \times \left( \sum_{n=0}^{\infty} S_n(-P)t^n \right),
\]

where

\[
S_n(A - P) = \sum_{j=0}^{n} S_{n-j}(-P)S_j(A).
\]

**Definition 11.** [9] Given a function \( f \) on \( \mathbb{R}^n \), the divided difference operator is defined as follows

\[
\partial_{p_ip_{i+1}}(f) = \frac{f(p_1, \ldots, p_i, p_{i+1}, \ldots, p_n) - f(p_1, \ldots, p_{i-1}, p_{i+1}, p_i, p_{i+2} \ldots p_n)}{p_i - p_{i+1}}.
\]

**Definition 12.** Let \( n \) be a positive integer and \( P = \{p_1, p_2\} \) are sets of given variables, then, the \( n \)-th symmetric function \( S_n(p_1 + p_2) \) is defined by

\[
S_n(p_1 + p_2) = \frac{p_1^{n+1} - p_2^{n+1}}{p_1 - p_2},
\]

with

\[
S_0(P) = S_0(p_1 + p_2) = 1,
S_1(P) = S_1(p_1 + p_2) = p_1 + p_2,
S_2(P) = S_2(p_1 + p_2) = p_1^2 + p_1p_2 + p_2^2,
\]

\[\vdots\]

**Definition 13.** Given an alphabet \( P = \{p_1, p_2\} \), the symmetrizing operator \( \delta^k_{p_1, p_2} \) is defined by

\[
\delta^k_{p_1, p_2}(g) = \frac{p_1^k g(p_1) - p_2^k g(p_2)}{p_1 - p_2}, \text{ for all } k \in \mathbb{N}_0.
\]

3. Generating functions of the products of Tribonacci numbers and some well-known numbers

The following Proposition allows us to obtain many new generating functions of binary products of Tribonacci numbers, some well-known numbers and Chebychev polynomials.

**Proposition 1.** [11] Given two alphabets \( P = \{p_1, p_2\} \) and \( A = \{a_1, a_2, \ldots\} \), we have

\[
\sum_{n=0}^{\infty} S_n(A)S_{k+n-1}(p_1 + p_2)t^n = \frac{\sum_{n=0}^{k-1} S_n(-A)p_1^n p_2^n S_{k-n-1}(p_1 + p_2)t^n}{\prod_{a \in A} (1 - ap_1t) \prod_{a \in A} (1 - ap_2t)} - \frac{p_1^k p_2^k S_{k+k+1}(-A)S_n(p_1 + p_2)}{\prod_{a \in A} (1 - ap_1z) \prod_{a \in A} (1 - ap_2z)}.
\]
Lemma 2. Given two alphabets $A = \{a_1, a_2, a_3\}$ using the formula (2.1), we have

$$\sum_{n=0}^{\infty} S_n(A)t^n = \frac{1}{(1-a_1t)(1-a_2t)(1-a_3t)},$$

with

$$(1-a_1t)(1-a_2t)(1-a_3t) = 1 + S_1(-A)t + S_2(-A)t^2 + S_3(-A)t^3.$$ Setting $S_1(-A) = S_2(-A) = S_3(-A) = -1$ in (3.1) and this gives

$$\sum_{n=0}^{\infty} S_n(A)t^n = \frac{1}{1-t-t^2-t^3},$$

which represents a generating function for Tribonacci numbers, such that $T_n = S_n(A)$.

- The case of $A = \{a_1, a_2, a_3\}$ and $P = \{p_1, p_2\}$

If $k = 0, 1$ in Proposition 3.1 we deduce the following lemmas.

Lemma 1. Given two alphabets $A = \{a_1, a_2, a_3\}$ and $P = \{p_1, p_2\}$, we have

$$\sum_{n=0}^{\infty} S_n(A)S_n(P)t^n = \frac{1 - p_1p_2S_2(-A)t^2 - p_1p_2(p_1 + p_2)S_3(-A)t^3}{(1-a_1p_1t)(1-a_2p_1t)(1-a_3p_1t)(1-a_1p_2t)(1-a_2p_2t)(1-a_3p_2t)}.$$

Lemma 2. Given two alphabets $A = \{a_1, a_2, a_3\}$ and $P = \{p_1, p_2\}$, we have

$$\sum_{n=0}^{\infty} S_n(A)S_{n-1}(P)t^n = \frac{-S_1(-A)t - (p_1 + p_2)S_2(-A)t^2 - S_3(-A)((p_1 + p_2)^2 - p_1p_2)t^3}{(1-a_1p_1t)(1-a_2p_1t)(1-a_3p_1t)(1-a_1p_2t)(1-a_2p_2t)(1-a_3p_2t)}.$$

Replacing $p_2$ by $(-p_2)$ in (3.3) and (3.4), we obtain

$$\sum_{n=0}^{\infty} S_n(A)S_n(p_1 + [-p_2])t^n = \frac{1 + p_1p_2S_2(-A)t^2 + p_1p_2(p_1 - p_2)S_3(-A)t^3}{(1-a_1p_1t)(1-a_2p_1t)(1-a_3p_1t)(1+a_1p_2t)(1+a_2p_2t)(1+a_3p_2t)},$$

$$\sum_{n=0}^{\infty} S_n(A)S_{n-1}(p_1 + [-p_2])t^n = \frac{-S_1(-A)t - (p_1 - p_2)S_2(-A)t^2 - S_3(-A)((p_1 - p_2)^2 + p_1p_2)t^3}{(1-a_1p_1t)(1-a_2p_1t)(1-a_3p_1t)(1+a_1p_2t)(1+a_2p_2t)(1+a_3p_2t)},$$

with

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\[(1 - a_1 p_1 t) (1 - a_2 p_1 t) (1 - a_3 p_1 t) (1 + a_1 p_2 t) (1 + a_2 p_2 t) (1 + a_3 p_2 t) = 1 + (p_1 - p_2) S_1(-A) t + (S_2(-A) (p_1 - p_2)^2 - p_1 p_2 S_2(-A) (S_2(-A) - 2)) t^2 + [S_3(-A) (p_1 - p_2)^3 - p_1 p_2 (p_1 - p_2) (S_2(-A) S_1(-A) - 3S_3(-A))] t^3 + (-p_1 p_2 (p_1 - p_2)^2 S_3(-A) S_1(-A) + p_1^2 p_2^2 (S_2(-A)^2 - 2S_3(-A) S_1(-A))) t^4 + S_3(-A) p_1^2 p_2^2 (p_1 - p_2) S_2(-A) t^5 - S_3(-A)^2 p_1^3 p_2^3 t^6.\]

This case consists of four related parts.

Firstly, the substitutions of \(S_1(-A) = S_2(-A) = S_3(-A) = -1\) and \(\begin{cases} p_1 - p_2 = k, \\ p_1 p_2 = 1, \end{cases}\) in (3.5), we have the following theorems and corollary.

**Theorem 1.** For \(n \in \mathbb{N}\), the new generating function of the binary product of \(k\)-Fibonacci numbers with Tribonacci numbers is given by

\[
(3.7) \quad \sum_{n=0}^{\infty} T_n F_{k,n} t^n = \frac{1 - t^2 - kt^3}{1 - kt - (k^2 + 3) t^2 - k(k^2 + 4) t^3 - (k^2 + 1) t^4 + kt^5 - t^6}.
\]

We can state the following corollary.

**Corollary 1.** The following identity holds true:

\[
T_n F_{k,n} = S_n(A) S_n(p_1 + [-p_2]).
\]

**Theorem 2.** For \(n \in \mathbb{N}\), the new generating function for the combined Tribonacci numbers and \(k\)-Lucas numbers is given by

\[
(3.8) \quad \sum_{n=0}^{\infty} L_{k,n} T_n t^n = \frac{2 - kt - (k^2 + 2) t^2 - k(k^2 + 3) t^3}{1 - kt - (k^2 + 3) t^2 - k(k^2 + 4) t^3 - (k^2 + 1) t^4 + kt^5 - t^6}.
\]

**Proof.** We have

\[
L_{k,n} = 2S_n(p_1 + [-p_2]) - kS_{n-1}(p_1 + [-p_2]).\quad(see \ [3]),
\]
We see that
\[
\sum_{n=0}^{\infty} L_{k,n} T_n t^n = \sum_{n=0}^{\infty} (2S_n(p_1 + [-p_2]) - kS_{n-1}(p_1 + [-p_2]))S_n(A) t^n
\]
\[
= 2 \sum_{n=0}^{\infty} S_n(p_1 + [-p_2])S_n(A) t^n - k \sum_{n=0}^{\infty} S_{n-1}(p_1 + [-p_2])S_n(A) t^n
\]
\[
= 2 \sum_{n=0}^{\infty} F_{k,n} T_n t^n - k \sum_{n=0}^{\infty} S_{n-1}(p_1 + [-p_2])S_n(A) t^n
\]
\[
= 2 \sum_{n=0}^{\infty} F_{k,n} T_n t^n - k \sum_{n=0}^{\infty} (p_1^n - (-p_2)^n)S_n(A) t^n
\]
\[
= 2 \sum_{n=0}^{\infty} F_{k,n} T_n t^n - k \left( \sum_{n=0}^{\infty} S_n(A)(p_1 t)^n - \sum_{n=0}^{\infty} S_n(A)(-p_2 t)^n \right)
\]
\[
= 2 \sum_{n=0}^{\infty} F_{k,n} T_n t^n - k \left( \frac{1}{1 - p_1 t - p_1^2 t^2 - p_1^3 t^3} - \frac{1}{1 + p_2 t - p_2^2 t^2 + p_2^3 t^3} \right)
\]
\[
= 2 \sum_{n=0}^{\infty} F_{k,n} T_n t^n - k \left( \frac{t + (p_1 - p_2) t^2 + ((p_1 - p_2)^2 + p_1 p_2) t^3}{1 - (p_1 - p_2)t - ((p_1 - p_2)^2 + 3p_1 p_2)t^2 - ((p_1 - p_2)^3 + 4p_1 p_2(p_1 - p_2))t^3 - p_1 p_2((p_1 - p_2)^2 + p_1 p_2)t^4 + p_1^2 p_2^2(p_1 - p_2)t^5 - p_1^3 p_2^3 t^6} \right)
\]

Let’s remember that for k-Lucas numbers \( \{p_1 - p_2 = k, \ p_1 p_2 = 1, \} \) and then
\[
\sum_{n=0}^{\infty} L_{k,n} T_n t^n = \frac{2 - 2t^2 - 2kt^3}{1 - kt - (k^2 + 3)t^2 - k(k^2 + 4)t^3 - (k^2 + 1)t^4 + kt^5 - t^6}
\]
\[
- \frac{kt + k^2 t^2 + k(k^2 + 1)t^3}{1 - kt - (k^2 + 3)t^2 - k(k^2 + 4)t^3 - (k^2 + 1)t^4 + kt^5 - t^6}
\]
\[
= \frac{2 - kt - (k^2 + 2)t^2 - k(k^2 + 3)t^3}{1 - kt - (k^2 + 3)t^2 - k(k^2 + 4)t^3 - (k^2 + 1)t^4 + kt^5 - t^6}
\]

This completes the proof. \( \square \)

- Put \( k = 1 \) in the relationships (3.7) and (3.8), we obtain the following corollaries.

**Corollary 2.** For \( n \in \mathbb{N} \), the new generating function of the product of Fibonacci numbers and Tribonacci numbers is given by
\[
\sum_{n=0}^{\infty} T_n F_n t^n = \frac{1 - t^2 - t^3}{1 - t - 4t^2 - 5t^3 - 2t^4 + t^5 - t^6}
\]
**Corollary 3.** For \( n \in \mathbb{N} \), the new generating function of the product of Fibonacci numbers and Tribonacci numbers is given by

\[
\sum_{n=0}^{\infty} T_n L_n t^n = \frac{1 - t - 3t^2 - 4t^3}{1 - t - 4t^2 - 5t^3 - 2t^4 + t^5 - t^6}.
\]

Secondly, the substitution of \( S_1(-A) = S_2(-A) = S_3(-A) = -1 \) and \( \{ p_1 - p_2 = 2, \ p_1p_2 = k, \) in (3.6), we deduce the following theorems and corollary.

**Theorem 3.** For \( n \in \mathbb{N} \), the new generating function of the binary product of \( k \)-Pell numbers with Tribonacci numbers

\[
\sum_{n=0}^{\infty} T_n P_{k,n} t^n = t + 2t^2 + (k + 4)t^3
\]

\( \frac{1}{1 - 2t - (4 + 3k)t^2 - 8(1 + k)t^3 - k(4 + k)t^4 + 2k^2t^5 - k^3t^6} \)

We can state the following corollary.

**Corollary 4.** The following identity holds true:

\[
T_n P_{k,n} = S_n(A)S_{n-1}(p_1 + [-p_2]).
\]

**Theorem 4.** The generating function for the combined Tribonacci numbers and \( k \)-Pell-Lucas numbers is given by

\[
\sum_{n=0}^{\infty} Q_{k,n} T_n t^n = \frac{2 - 2t - 2(k + 2)t^2 - 2(3k + 4)t^3}{1 - 2t - (4 + 3k)t^2 - 8(1 + k)t^3 - k(4 + k)t^4 + 2k^2t^5 - k^3t^6}.
\]

**Proof.** We have

\[
Q_{k,n} = 2S_n(p_1 + [-p_2]) - 2S_{n-1}(p_1 + [-p_2]), \text{ (see [3]).}
\]
We see that

\[
\sum_{n=0}^{\infty} Q_{k,n} T_n t^n = \sum_{n=0}^{\infty} (2S_n(p_1 + [-p_2]) - 2S_{n-1}(p_1 + [-p_2])) S_n(A) t^n
\]

\[= 2 \sum_{n=0}^{\infty} S_n(p_1 + [-p_2]) S_n(A) t^n - 2 \sum_{n=0}^{\infty} S_{n-1}(p_1 + [-p_2]) S_n(A) t^n
\]

\[= \frac{2}{p_1 + p_2} \left( p_1 \sum_{n=0}^{\infty} S_n(A)(p_1 t)^n + p_2 \sum_{n=0}^{\infty} S_n(A)(-p_2 t)^n \right) - 2 \sum_{n=0}^{\infty} S_{n-1}(p_1 + [-p_2]) S_n(A) t^n
\]

\[= \frac{2}{p_1 + p_2} \left( \frac{p_1}{1 - p_1 t - p_1^2 t^2 - p_1^3 t^3} + \frac{p_2}{1 + p_2 t - p_2^2 t^2 + p_2^3 t^3} \right)
\]

\[-2 \sum_{n=0}^{\infty} S_{n-1}(p_1 + [-p_2]) S_n(A) t^n
\]

We have

\[
\frac{p_1}{1 - p_1 t - p_1^2 t^2 - p_1^3 t^3} + \frac{p_2}{1 + p_2 t - p_2^2 t^2 + p_2^3 t^3}
\]

\[
= \frac{p_1 + p_2 - p_1 p_2 (p_1 + p_2) t^2 - p_1 p_2 (p_1^2 - p_2^2) t^3}{1 - (p_1 - p_2) t - (p_1^2 - p_2^2) t^2 - ((p_1^3 - p_2^3) + p_1^2 p_2 - p_1 p_2^2) t^3 - p_1 p_2 (p_1^2 - p_2^2) t^4 + p_1^2 p_2 (p_1 - p_2) t^5 - p_1^3 p_2 t^6}
\]

So

\[
\sum_{n=0}^{\infty} Q_{k,n} T_n t^n = 2 \left( \frac{1 - p_1 p_2 t^2 - p_1 p_2 (p_1 - p_2) t^3}{1 - (p_1 - p_2) t - ((p_1 - p_2)^2 + 3 p_1 p_2) t^2 - ((p_1 - p_2)^3 + 4 p_1 p_2 (p_1 - p_2)) t^3 - p_1 p_2 ((p_1 - p_2)^2 + p_1 p_2) t^4 + p_1^2 p_2 (p_1 - p_2) t^5 - p_1^3 p_2 t^6} \right)
\]

\[-2 \sum_{n=0}^{\infty} S_{n-1}(p_1 + [-p_2]) S_n(A) t^n
\]
Let’s remember that for k-Lucas numbers \( \left\{ \begin{array}{l}
p_1 - p_2 = 2, \\
p_1p_2 = k,
\end{array} \right. \) and then

\[
\sum_{n=0}^{\infty} Q_{k,n} T_n t^n = \frac{1 - kt^2 - 2kt^3}{1 - 2t - (4 + 3k)t^2 - 8(1 + k)t^3 - k(4 + k)t^4 + 2k^2t^5 - k^3t^6} \\
\quad \cdot \frac{1}{t + 2t^2 + (k + 4)t^3} \\
\quad - \frac{2}{1 - 2t - (4 + 3k)t^2 - 8(1 + k)t^3 - k(4 + k)t^4 + 2k^2t^5 - k^3t^6} \\
\quad \cdot \frac{2 - 2t - 2(k + 2)t^2 - 2(3k + 4)t^3}{1 - 2t - (4 + 3k)t^2 - 8(1 + k)t^3 - k(4 + k)t^4 + 2k^2t^5 - k^3t^6}.
\]

This completes the proof. \( \square \)

- Put \( k = 1 \) in the relationships (3.9) and (3.10), we obtain the following corollaries.

**Corollary 5.** For \( n \in \mathbb{N} \), the new generating function of the binary product of Pell numbers and Tribonacci numbers is given by

\[
\sum_{n=0}^{\infty} T_n P_n t^n = \frac{t + 2t^2 + 5t^3}{1 - 2t - 7t^2 - 16t^3 - 5t^4 + 2t^5 - t^6}.
\]

**Corollary 6.** For \( n \in \mathbb{N} \), the new generating function of the binary product of Pell numbers and Tribonacci numbers is given by

\[
\sum_{n=0}^{\infty} T_n Q_n t^n = \frac{2 - 2t - 6t^2 - 17t^3}{1 - 2t - 7t^2 - 16t^3 - 5t^4 + 2t^5 - t^6}.
\]

**Third**, the substitution, \( S_1(-A) = S_2(-A) = S_3(-A) = -1 \) and \( \left\{ \begin{array}{l}
p_1 - p_2 = 3k, \\
p_1p_2 = -2,
\end{array} \right. \) in (3.6) we deduce the following theorem.

**Theorem 5.** For \( n \in \mathbb{N} \), the new generating function of the binary product of k-Mersenne numbers and Tribonacci numbers is given by

(3.11)

\[
\sum_{n=0}^{\infty} T_n M_{k,n} t^n = \frac{t + 3kt^2 + (9k^2 - 2)t^3}{1 - 3kt - 3(3k^2 - 2)t^2 - 3k(9k^2 - 8)t^3 + 2(9k^2 - 2)t^4 + 12kt^5 + 8t^6}.
\]

We can state the following corollary.

**Corollary 7.** The following identity holds true:

\[ T_n M_{k,n} = S_n(A) S_{n-1}(p_1 + [-p_2]). \]

- Put \( k = 1 \) in the relationship (3.11), we obtain the following corollaries.

**Corollary 8.** For \( n \in \mathbb{N} \), the new generating function of the binary product of Mersenne numbers and Tribonacci numbers is given by

\[
\sum_{n=0}^{\infty} T_n M_n t^n = \frac{t + 3t^2 + 7t^3}{1 - 3t - 3t^2 - 3t^3 + 14t^4 + 12t^5 + 8t^6}.
\]
Fourthly, the substitution of $S_1(-A) = S_2(-A) = S_3(-A) = -1$ and \( \begin{cases} p_1 - p_2 = 2, \\ p_1 p_2 = k, \end{cases} \) in (3.6), we deduce the following theorems and corollary.

For \( n \in \mathbb{N} \), the new generating function of the binary product of \( k \)-Jacobsthal numbers with Tribonacci numbers

\[
\sum_{n=0}^{\infty} T_n J_{k,n} t^n = \frac{t + kt^2 + (k^2 + 2)t^3}{1 - kt - (k^2 + 6)t^2 - k(k^2 + 8)t^3 + 2(k^2 + 2)t^4 + 4kt^5 - 8t^6},
\]

We can state the following corollary.

**Corollary 9.** The following identity holds true:

\[
T_n J_{k,n} = S_n(A)S_{n-1}(p_1 + [-p_2]).
\]

**Theorem 6.** For \( n \in \mathbb{N} \), the new generating function for the combined Tribonacci numbers and \( k \)-Jacobsthal-Lucas numbers is given by

\[
\sum_{n=0}^{\infty} T_n j_{k,n} t^n = \frac{2 - kt - (k^2 + 4)t^2 - k(k^2 + 6)t^3}{1 - kt - (k^2 + 6)t^2 - k(k^2 + 8)t^3 + 2(k^2 + 2)t^4 + 4kt^5 - 8t^6},
\]

**Proof.** We have

\[
j_{k,n} = 2S_n(p_1 + [-p_2]) - kS_{n-1}(p_1 + [-p_2]), \text{ (see [3]).}
\]
We see that
\[ \sum_{n=0}^{\infty} T_{n}j_{k,n}t^{n} = \sum_{n=0}^{\infty} (2S_{n}(p_{1} + [-p_{2}]) - kS_{n-1}(p_{1} + [-p_{2}])S_{n}(A)t^{n} \]
\[ = 2 \sum_{n=0}^{\infty} S_{n}(p_{1} + [-p_{2}])S_{n}(A)t^{n} - k \sum_{n=0}^{\infty} S_{n-1}(p_{1} + [-p_{2}])S_{n}(A)t^{n} \]
\[ = \frac{2}{p_{1} + p_{2}} \sum_{n=0}^{\infty} (p_{1}^{n+1} - (-p_{2})^{n+1})S_{n}(A)t^{n} - k \sum_{n=0}^{\infty} T_{n}j_{k,n}t^{n} \]
\[ = \frac{2}{p_{1} + p_{2}} \left( \frac{p_{1}}{1 - p_{1}t} - \frac{p_{2}}{1 + p_{2}t^{2} - p_{2}^{2}t^{3}} \right) - k \sum_{n=0}^{\infty} T_{n}j_{k,n}t^{n} \]
\[ = 2 \left( \frac{1 - p_{1}p_{2}t^{2} - p_{1}p_{2}(p_{1} - p_{2})t^{3}}{1 - (p_{1} - p_{2})t - ((p_{1} - p_{2})^{2} + 3p_{1}p_{2})t^{2} - ((p_{1} - p_{2})^{3} + 4p_{1}p_{2}(p_{1} - p_{2}))t^{3}} - p_{1}p_{2}((p_{1} - p_{2})^{2} + p_{1}p_{2})t^{4} + p_{1}^{2}p_{2}^{2}(p_{1} - p_{2})t^{5} - p_{1}^{3}p_{2}^{3}t^{6} \right) \]
\[ - k \sum_{n=0}^{\infty} J_{n}p_{n}t^{n} \]
\[ = \frac{2 - 4t^{2} - 4kt^{3}}{1 - kt - (k^{2} + 6)t^{2} - k(k^{2} + 8)t^{3} - 2(k^{2} + 2)t^{4} + 4kt^{5} - 8t^{6}} - \frac{kt + k^{2}t^{2} + k(k^{2} + 2)t^{3}}{1 - kt - (k^{2} + 6)t^{2} - k(k^{2} + 8)t^{3} - 2(k^{2} + 2)t^{4} + 4kt^{5} - 8t^{6}} \]
\[ = 2 - kt - (k^{2} + 4)t^{2} - k(k^{2} + 6)t^{3} \]
\[ = \frac{2 - kt - (k^{2} + 6)t^{2} - k(k^{2} + 8)t^{3} - 2(k^{2} + 2)t^{4} + 4kt^{5} - 8t^{6}}{1 - kt - (k^{2} + 6)t^{2} - k(k^{2} + 8)t^{3} - 2(k^{2} + 2)t^{4} + 4kt^{5} - 8t^{6}}. \]

This completes the proof. \( \square \)

- Put \( k = 1 \) in the relationships (3.12) and (3.13), we obtain the following corollaries.

**Corollary 10.** For \( n \in \mathbb{N} \), the new generating function of the binary product of Jacobsthal numbers and Tribonacci numbers is given by
\[ \sum_{n=0}^{\infty} T_{n}j_{n}t^{n} = \frac{t + t^{2} + 3t^{3}}{1 - t - 7t^{2} - 9t^{3} - 6t^{4} + 4t^{5} - 8t^{6}}. \]

**Corollary 11.** For \( n \in \mathbb{N} \), the new generating function of the binary product of Jacobsthal-Lucas numbers and Tribonacci numbers is given by
\[ \sum_{n=0}^{\infty} T_{n}j_{n}t^{n} = \frac{2 - t - 5t^{2} - 7t^{3}}{1 - t - 7t^{2} - 9t^{3} - 6t^{4} + 4t^{5} - 8t^{6}}. \]
4. Generating Functions of Binary Products of Tribonacci Numbers and Chebychev Polynomials

In this part, we now derive the new generating functions of binary product of Tribonacci numbers with Chebychev polynomials.

Replacing \( p_1 \) by \( 2p_1 \) and \( p_2 \) by \((-2p_2)\) in (3.3) yields

\[
(4.1) \quad \sum_{n=0}^{\infty} S_n(A)S_n(2p_1 + [-2p_2])t^n = \frac{1 + 4p_1p_2S_2(-A)t^2 + 8p_1p_2(p_1 - p_2)S_3(-A)t^3}{P_{AP}},
\]

with

\[
P_{AP} = (1 - 2a_1p_1t)(1 - 2a_2p_1t)(1 - 2a_3p_1t)(1 + 2a_1p_2t)(1 + 2a_2p_2t)(1 + 2a_3p_2t)
= 1 + 2(p_1 - p_2)S_1(-A)t + 4\left(S_2(-A)(p_1 - p_2)^2 - p_1p_2(S_1(-A)^2 - 2S_2(-A))\right)t^2
- 8\left[-S_3(-A)(p_1 - p_2)^3 + p_1p_2(p_1 - p_2)(S_2(-A)S_1(-A) - 3S_3(-A))\right]t^3
+ 16\left[-p_1p_2(p_1 - p_2)S_3(-A)S_1(-A) + p_1^2p_2^2(S_2(-A)^2 - 2S_3(-A)S_1(-A))\right]t^4
+ 32S_3(-A)p_1^2p_2^2(p_1 - p_2)S_2(-A)t^5 - 64S_3(-A)^2p_1^3p_2^3t^6.
\]

Remark 2. In what follows, \( \{T_n(x)\}_{n \in \mathbb{N}} \) (respectively \( \{U_n(x)\}_{n \in \mathbb{N}} \)) represents the Chebychev polynomial sequence of the first (respectively second) order and \( \{T_n\}_{n \in \mathbb{N}} \) the Tribonacci sequence. \( P = \{p_1, p_2\} \) (respectively \( A = \{a_1, a_2, a_3\} \)) is the set of roots of characteristic equation of Chebychev polynomial of first and second order (respectively of Tribonacci numbers).

By making the following restrictions: \( S_1(-A) = S_2(-A) = S_3(-A) = -1 \) and \( 4p_1p_2 = -1, \), in (4.1), we get following Theorems:

**Theorem 7.** For \( n \in \mathbb{N} \), the new generating function for the combined Tribonacci numbers and Chebychev polynomials of second kind is given by

\[
(4.2) \quad \sum_{n=0}^{\infty} T_nU_n(x)t^n = \frac{1 + t^2 + 2xt^3}{1 - 2xt - (4x^2 - 3)t^2 - 8x(x^2 - 1)t^3 + (4x^2 - 1)t^4 + 2xt^5 + t^6},
\]

**Corollary 12.** The following identity holds true:

\[
T_nU_n(x) = S_n(A)S_n(2p_1 + [-2p_2]).
\]

**Theorem 8.** For \( n \in \mathbb{N} \), the new generating function of the binary product of Tribonacci numbers and Chebychev polynomials of first kind is given by

\[
(4.3) \quad \sum_{n=0}^{\infty} T_nT_n(x)t^n = \frac{1 - xt - (2x^2 - 1)t^2 - x(4x^2 - 3)t^3}{1 - 2xt - (4x^2 - 3)t^2 - 8x(x^2 - 1)t^3 + (4x^2 - 1)t^4 + 2xt^5 + t^6}.
\]

**Proof.** We have

\[
T_n(x) = S_n(2p_1 + [-2p_2]) - xS_{n-1}(2p_1 + [-2p_2]), \quad \text{(see [13])}.
\]

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We see that
\[
\sum_{n=0}^{\infty} T_n T_n (x) t^n = \sum_{n=0}^{\infty} S_n(A) (S_n (2p_1 + [-2p_2]) - x S_n (2p_1 + [-2p_2])) t^n \\
= \sum_{n=0}^{\infty} S_n(A) S_n (2p_1 + [-2p_2]) t^n - x \sum_{n=0}^{\infty} S_n(A) S_n (2p_1 + [-2p_2]) t^n \\
= \sum_{n=0}^{\infty} T_n U_n(x) t^n - \frac{x}{2(p_1 + p_2)} \sum_{n=0}^{\infty} S_n(A) ((2p_1)^n - (-2p_2)^n) t^n \\
= \sum_{n=0}^{\infty} T_n U_n(x) t^n - \frac{x}{2(p_1 + p_2)} \left( \sum_{n=0}^{\infty} S_n(A) (2p_1 t)^n - \sum_{n=0}^{\infty} S_n(A) (-2p_2 t)^n \right).
\]

Since
\[
\sum_{n=0}^{\infty} S_n(A) t^n = \frac{1}{1 - t - t^2 - t^3},
\]
we have
\[
\sum_{n=0}^{\infty} T_n T_n (x) t^n = \sum_{n=0}^{\infty} T_n U_n(x) t^n \\
- \frac{x}{2(p_1 + p_2)} \left( \frac{1}{1 - 2p_1 t - 4p_1^2 t^2 - 8p_1^3 t^3} - \frac{1}{1 + 2p_2 t - 4p_2^2 t^2 + 8p_2^3 t^3} \right) \\
= \sum_{n=0}^{\infty} T_n U_n(x) t^n \\
- \frac{x}{2} \left( \frac{2t + 4(p_1 - p_2) t^2 + 2(4(p_1 - p_2)^2 + 4p_1 p_2) t^3}{1 - 2xt - (4x^2 - 3) t^2 - 8x(x^2 - 1) t^3 + (4x^2 - 1) t^4 + 2xt^5 + t^6} \right) \\
= \sum_{n=0}^{\infty} T_n U_n(x) t^n \\
- x \left( \frac{t + 2xt^2 + (4x^2 - 1) t^3}{1 - 2xt - (4x^2 - 3) t^2 - 8x(x^2 - 1) t^3 + (4x^2 - 1) t^4 + 2xt^5 + t^6} \right),
\]

Thus
\[
\sum_{n=0}^{\infty} T_n T_n (x) t^n = \frac{1 + t^2 + 2xt^3}{1 - 2xt - (4x^2 - 3) t^2 - 8x(x^2 - 1) t^3 + (4x^2 - 1) t^4 + 2xt^5 + t^6} \\
- \frac{xt + 2x^2 t^2 + x(4x^2 - 1) t^3}{1 - 2xt - (4x^2 - 3) t^2 - 8x(x^2 - 1) t^3 + (4x^2 - 1) t^4 + 2xt^5 + t^6}.
\]
therefore
\[ \sum_{n=0}^{\infty} T_n T_n (x) t^n = \frac{1 - xt - (2x^2 - 1) t^2 - x (4x^2 - 3) t^3}{1 - 2xt - (4x^2 - 3)t^2 - 8x(x^2 - 1)t^3 + (4x^2 - 1)t^4 + 2xt^5 + t^6}. \]

This completes the proof. \(\square\)

5. Conclusion

In this paper, by making use of Eq. (3.1), we have derived some new generating functions of binary products of Tribonacci numbers and well-known Numbers and Chebychev polynomials. The derived lemmas and corollaries are based on symmetric functions and products of these numbers and polynomials.

References


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