INFINITE LOG-CONVEXITY

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ABSTRACT. A criteria to verify log-convexity of sequences is presented. Iterating this criteria produces infinitely log-convex sequences. As an application, several classical examples of sequences arising in Combinatorics and Special Functions are presented. The paper concludes with a conjecture regarding coefficients of chromatic polynomials.

1. INTRODUCTION

Questions about the ordering of a sequence of non-negative real numbers $\mathtt{a} = \{a_k\}_k$, for $0 \leq k \leq n$, have appeared in the literature since Newton. He established that if $P(x)$ is a polynomial, all of whose zeros are real and negative, then the sequence of its coefficients $\mathtt{a} = \{a_k\}_k$ is log-concave; that is, $a_k^2 - a_{k-1}a_{k+1} \geq 0$ for $1 \leq k \leq n-1$. A weaker condition on sequences is that of unimodality: that is, there is an index r such that $a_0 \leq a_1 \leq \cdots \leq a_r \geq a_{r+1} \geq \cdots \geq a_n$. An elementary argument shows that a log-concave sequence must be unimodal. A sequence $\mathtt{a} = \{a_k\}_k$ is called log-convex if $a_k^2 - a_{k-1}a_{k+1} \leq 0$ for $1 \leq k \leq n-1$.

These concepts can be expressed in terms of the operator $\mathtt{a} \mapsto \mathcal{L}(\mathtt{a})$ defined by $\mathcal{L}(\mathtt{a})_k = a_k^2 - a_{k-1}a_{k+1}$. In this notation, the sequence $\mathtt{a} = \{a_k\}_k$ is log-concave if it satisfies $\mathcal{L}(\mathtt{a})_k \geq 0$ for $k \geq 1$. Similarly, the sequence is log-convex if $\mathcal{L}(\mathtt{a})_k \leq 0$. Iteration of $\mathcal{L}$ leads to the notion of ℓ-log-concave sequences, defined by the property that the sequences $\mathcal{L}^j(\mathtt{a})$ are all non-positive for $1 \leq j \leq \ell$ and $\mathtt{a}$ is infinitely log-convex if it is ℓ-log-convex for every $\ell \in \mathbb{N}$. The definitions of ℓ-log-concave and infinitely log-concave are similar.

The results presented here originate with the sequence of coefficients $\{d_i(n)\}_i$ of the polynomial

$$P_n(a) = \sum_{i=0}^{n} d_i(n)a^i, \tag{1.1}$$

defined by

$$d_i(n) = 2^{-2n}\sum_{k=i}^{n} 2^k \binom{2n-2k}{n-k}\binom{n+k}{n}\binom{k}{i}. \tag{1.2}$$

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This polynomial appears in the evaluation of a definite integral. More details are presented in Section 5.

The goal of the present work is to develop a criteria which verifies the log-convexity of a variety of classical sequences. We record an elementary observation of independent interest.

Lemma 1. *A positive sequence* $\mathtt{a} = \{a_k\}_k$ *is log-convex if and only if* $\mathtt{a}^{-1} = \{1/a_k\}_k$ *is log-concave.*

Proof. Simply observe that

$$\mathcal{L}\left(\frac{1}{a_k}\right) = \frac{1}{a_{k-1}a_{k+1}} - \frac{1}{a_k^2} = \frac{\mathcal{L}(\mathtt{a})_k}{a_{k-1}a_k^2a_{k+1}}. \tag{1.3}$$

□

Remark 1. This does not extend to k-log-concavity for $k \geq 2$. For instance, the sequence $\{1, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{31}\}$ is 2-log-convex but the sequence of reciprocals is not 2-log-concave.

2. The criteria

In this section we establish the basic criteria used to establish infinite log-convexity of sequences.

Proposition 1. *Let* $\mathtt{a} = \{a_k\}_k$*, with* $a_k = \int_X f^k(x)\, d\mu(x)$ *for a certain positive function* f *on a measure space* (X, μ)*. Then* $\mathtt{a} = \{a_k\}_k$ *is infinitely log-convex.*

Proof. It suffices to prove that $\mathcal{L}(\mathtt{a})_k \leq 0$. The general statement follows by iteration of the argument. The initial step is a consequence of

$$\begin{aligned} -\mathcal{L}(\mathtt{a})_k &= a_{k-1}a_{k+1} - a_k^2 \\ &= \int_{X\times X} f^{k-1}(x)f^{k+1}(y)d\mu(x)d\mu(y) - \int_{X\times X} f^k(x)f^k(y)d\mu(x)d\mu(y) \\ &= \frac{1}{2}\int_{X\times X} f^k(x)f^k(y)\left(\frac{f(x)}{f(y)} + \frac{f(y)}{f(x)} - 2\right)d\mu(x)d\mu(y) \\ &= \frac{1}{2}\int_{X\times X} f^{k-1}(x)f^{k-1}(y)(f(x)-f(y))^2\, d\mu(x)d\mu(y). \end{aligned}$$

To iterate this argument, observe that $\mathcal{L}\mathtt{a}$ also satisfies the hypothesis of this proposition. □

3. Examples of combinatorial sequences

This section presents a list of examples of log-convex sequences using Proposition 1.

Example 2. The central binomial coefficients $\left\{\binom{2k}{k}\right\}_k$ are infinitely log-convex.

Proof. This follows directly from Wallis' formula [6, Theorem 6.4.1] written in the form

$$\binom{2k}{k} = \frac{2}{\pi}\int_0^{\pi/2} (2\sin x)^{2k}\,dx. \tag{3.1}$$

□

Example 3. The Catalan numbers $C_k = \frac{1}{k+1}\binom{2k}{k}$ are infinitely log-convex.

Proof. Applying the Wallis' integral formula for $\binom{2k}{k}$ in (3.1), we obtain

$$C_k = \frac{2}{\pi}\int_0^{\pi/2}\int_0^1 \left(4t\sin^2 x\right)^k dx\,dt. \tag{3.2}$$

□

Example 4. Let $\{F_k\}_k$ be the sequence of Fibonacci numbers. Then $\{F_{2k}/k\}$ is infinitely log-convex.

Proof. This follows from the integral representation [11, eqn. (10.2)] written in the form

$$\frac{F_{2k}}{k} = \frac{1}{2}\int_0^{\pi}\left(\frac{3}{2}+\frac{\sqrt{5}}{3}\cos x\right)^{k-1} d\mu(x) \quad \text{with } d\mu(x) = \sin x\,dx. \tag{3.3}$$

□

Example 5. The reciprocals of the binomial coefficients $\mathtt{a_{row}} = \{\binom{n}{k}^{-1}\}_k$ form an infinitely log-concave sequence. The same holds for the sequence $\mathtt{a_{col}} = \{\binom{n}{k}^{-1}\}_n$.

Proof. Fix n and consider the expression $a_k = \binom{n}{k}^{-1}$. Using the integral representation of Euler's beta function

$$B(n,m) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)} = \int_0^1 t^{n-1}(1-t)^{m-1}dt,$$

we have

$$a_k = \int_0^1 \left(\frac{x}{1-x}\right)^k d\mu(k) \quad \text{with } d\mu(x) = (n+1)(1-x)^n\,dx. \tag{3.4}$$

Proposition 1 and (3.4) yield the infinite log-convexity of $\mathtt{a_{row}} = \{a_k\}_k$.

The second assertion follows from the representation

$$\binom{n}{k}^{-1} = \int_0^1 (n+1)(1-x)^n\,d\eta(x) \quad \text{with } d\eta(x) = \left(\frac{x}{1-x}\right)^k dx. \tag{3.5}$$

□

Example 6. The derangement sequence d_k is defined as the number of permutations in $\mathfrak{S}_k$ without fixed points. The integral representation of the even-indexed subsequence d_{2k} [17, page 313]

$$d_{2k} = \int_0^{\infty} (x-1)^{2k}\,d\mu(x) \quad \text{with} \quad d\mu(x) = e^{-x}dx \tag{3.6}$$

shows that $\{d_{2k}\}_k$ is infinitely log-convex.

Example 7. A permutation $\pi = \pi_1\pi_2\ldots\pi_n$ in the symmetric group $\mathfrak{S}_n$ is called alternating if its entries alternately rise or descend. The Euler number E_n counts the number of alternating permutations in $\mathfrak{S}_n$. Since $E_{2k} = (-1)^k\widetilde{E}_{2k}$ ($\widetilde{E}_{2k}$ denotes the Eulerian numbers) and by the integral representation of $\widetilde{E}_{2k}$ (see [3, eqn. (1)]), we have

$$E_{2k} = \frac{2}{\pi}\int_0^\infty \left(\frac{2\log x}{\pi}\right)^{2k} d\mu(x) \quad \text{with} \;\; d\mu(x) = \frac{dx}{1+x^2}. \tag{3.7}$$

Proposition 1 and (3.7) together imply that $\{E_{2k}\}_k$ is infinitely log-convex.

Example 8. The large Schröder numbers S_k count the number of paths on a $k\times k$ grid from the southwest corner $(0,0)$ to the northeast corner (k,k) using only single steps north, northeast or east that do not rise above the southwest-northeast diagonal. Proposition 1 and the integral representation (see [19, eqn. (1.10)])

$$S_k = \frac{1}{2\pi}\int_{3-2\sqrt{2}}^{3+2\sqrt{2}} \frac{1}{x^{k+2}}\,d\mu(x) \quad \text{with} \quad d\mu(x) = \sqrt{-x^2+6x-1}\,dx \tag{3.8}$$

show that $\{S_k\}_k$ is infinitely log-convex.

Example 9. The Motzkin numbers M_k count the number of lattice paths from $(0,0)$ to (k,k), consisting of steps $(0,2)$, $(2,0)$ and $(1,1)$ subject to never rising above the diagonal $y = x$. Apply the integral representation [15, Corollary 12 (e)]

$$M_{2k} = \frac{2}{\pi}\int_0^\pi (1+2\cos x)^{2k}\,d\mu(x) \quad \text{with } d\mu(x) = \sin^2 x\,dx \tag{3.9}$$

reveals that the even-indexed Motzkin sequence $\{M_{2k}\}_k$ is indeed infinitely log-convex.

Example 10. Let h_k be the number of lattice paths from $(0,0)$ to $(2k,0)$ with steps $(1,1)$, $(1,-1)$ and $(2,0)$, never falling below the x-axis and with no peaks at odd level. These numbers also count the number of sets of all tree-like polyhexes with $k+1$ hexagons. This is sequence `A002212` in OEIS. The integral representation

$$h_k = \frac{1}{2\pi}\int_1^5 x^{k-1} d\mu(x) \quad \text{with } d\mu(x) = \sqrt{(x-1)(5-x)}\,dx \tag{3.10}$$

and Proposition 1 show that $\{h_k\}_k$ is infinitely log-convex.

Example 11. Let w_k be the number of walks on a cubic lattice with k steps, starting and finishing on the xy-plane conditioned to never going below it. This is sequence `A005572` in OEIS. These numbers have the integral representation

$$w_k = \frac{1}{2\pi}\int_2^6 x^k d\mu(x) \quad \text{with} \;\; d\mu(x) = \sqrt{4-(4-x)^2}. \tag{3.11}$$

The usual argument shows that $\{h_k\}_k$ is infinitely log-convex.

Example 12. The central Delanoy numbers D_k enumerate the number of *king walks* on a $k \times k$ grid, from the $(0,0)$ corner to the upper right corner (k,k). The integral representation due to F. Qi et. al. [20, Theorem 1.3]

$$D_k = \frac{1}{\pi}\int_{3-2\sqrt{2}}^{3+2\sqrt{2}} \frac{1}{x^{k+1}}\, d\mu(x) \quad \text{with } d\mu(x) = \frac{dx}{\sqrt{-x^2+6x-1}} \tag{3.12}$$

shows that $\{D_k\}_k$ is infinitely log-convex.

Example 13. The Narayana numbers $N(n,k)$ count the number of lattice paths from $(0,0)$ to $(2n,0)$, with k peaks, not straying below the x-axis and using northeast and southeast steps. Applying the integral formula for the Euler's beta function, the infinite log-convexity of the reciprocals of $N(n,k) = \frac{1}{n}\binom{n}{k-1}\binom{n}{k}$ follows from the integral representation

$$\frac{1}{N(n,k)} = \int_0^1\int_0^1 \left(\frac{x}{1-x}\right)^k \left(\frac{y}{1-y}\right)^{k-1} d\mu(x,y), \tag{3.13}$$

where $d\mu(x,y) = n(n+1)^2(1-x)^n(1-y)^n\, dx\, dy$.

4. A variety of examples from special functions

This section presents a selection of sequences related to classical special functions.

Example 14. The sequence of factorials $\{k!\}_k$ is infinitely log-convex.

Proof. Apply the representation

$$k! = \int_0^\infty x^k\, d\mu(x) \quad \text{with} \quad d\mu(x) = e^{-x}dx. \tag{4.1}$$

□

Example 15. The classical Eulerian gamma and beta functions are defined by integral representations

$$\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}\, dt \tag{4.2}$$

and

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\, dt. \tag{4.3}$$

Specialization of these formulas and Proposition 1 give infinitely log-convex sequences. Example 14 corresponds to the special value $\Gamma(k+1) = k!$. Another infinitely log-convex sequence arising in this manner is $\{a_k\}_k$, with

$$a_k = \frac{(2k)!}{2^{2k}k!} = \frac{1}{\sqrt{\pi}}\Gamma\left(k+\tfrac{1}{2}\right). \tag{4.4}$$

Naturally, the specialization of (4.3) gives a double-indexed log-convex sequence (symmetric in m and n)

$$B(n,m) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)} = \frac{(n-1)!(m-1)!}{(n+m-1)!}. \tag{4.5}$$

Clearly, many other examples can be produced in this manner.

Example 16. The integral representation of the Riemann zeta function (see [21, eqn. (2.4.1)])

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}\,dx}{e^x - 1} \qquad \text{with } \mathrm{Re}(s) > 1 \tag{4.6}$$

gives for $k \in \mathbb{N}$,

$$\Gamma(k)\zeta(k) = \int_0^\infty x^k d\mu(x) \quad \text{with } d\mu(x) = \frac{dx}{x(e^x-1)}. \tag{4.7}$$

Proposition 1 shows that the sequence $\{\Gamma(k)\zeta(k)\}_k$ is infinitely log-convex.

Example 17. The values of the Riemann zeta function at even integers is given in terms of the Bernoulli numbers B_{2k} defined by the generating function

$$\coth x = \frac{1}{x}\sum_{k=0}^{\infty} \frac{B_{2k}}{(2k)!}(2x)^{2k}. \tag{4.8}$$

The aforementioned relation and taking the logarithmic derivative of the function $\sin z = z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2\pi^2}\right)$ with the substitution $z \mapsto ix$, it follows that

$$B_{2k} = \frac{(-1)^{k+1}2(2k)!}{(2\pi)^{2k}}\zeta(2k). \tag{4.9}$$

The integral representation (4.6) yields

$$\frac{B_{4k+2}}{4k+2} = \int_0^\infty 2\left(\frac{x}{2\pi}\right)^{4k+2} d\mu(x) \quad \text{with } d\mu(x) = \frac{dx}{x(e^x-1)}. \tag{4.10}$$

From here it follows that the sequence $\left\{\frac{1}{4k+2}B_{4k+2}\right\}_k$ is infinitely log-convex.

The next example emerges from a multi-dimensional integral:

Example 18. Fix $d \in \mathbb{N}$. Then the sequence $\left\{\frac{1}{(k+1)^d}\right\}_k$ is infinitely log-convex.

Proof. Apply the representation

$$\frac{1}{(k+1)^d} = \int_0^1 \cdots \int_0^1 (x_1x_2\cdots x_d)^k\, d\mu(\mathbf{x}) \tag{4.11}$$

with $d\mu(\mathbf{x}) = dx_1dx_2\cdots dx_d$. ∎

The final example in this section is a generalization of Example 3.

Example 19. The generating function of the Catalan numbers C_k is

$$G(x) = \frac{2}{1+\sqrt{1-4x}} = \sum_{k=0}^{\infty} C_k x^k. \tag{4.12}$$

Li et al. [14, eqn. (1.10)] considered the function

$$G_{a,b}(x) = \frac{1}{a+\sqrt{b-x}} = \sum_{k=0}^{\infty} \mathcal{C}_k(a,b) x^k \tag{4.13}$$

as a generalization of (4.12). The coefficients $\mathcal{C}_k(a,b)$ admit the integral representation [14, Theorem 3.1, eqn. (3.2)]

$$\mathcal{C}_k(a,b) = \frac{2}{\pi} \int_0^{\infty} \frac{s^2\, ds}{(a^2+s^2)(b+s^2)^{n+1}}, \tag{4.14}$$

(see [18, 7.4.1]. Proposition 1 shows that, for fixed a and b, the sequence $\{\mathcal{C}_k(a,b)\}_k$ is infinitely log-convex.

Among the expressions for $\mathcal{C}_n(a,b)$ one finds the finite sum [14, Theorem 2.1]

$$\mathcal{C}_n(a,b) = \frac{1}{(2n)!!b^{n+1/2}} \sum_{k=0}^{n} \binom{2n-k-1}{2(n-k)} \frac{k![2(n-k)-1]!!}{(1+a/\sqrt{b})^{k+1}}, \tag{4.15}$$

the hypergeometric representation

$$\mathcal{C}_n(a,b) = C_n \frac{\pi}{(2\sqrt{b})^n} \frac{1}{(a+\sqrt{b})^{n+1}} {}_2F_1\left(\left. \begin{matrix} 1-n \quad n \\ n+2 \end{matrix} \right| \frac{\sqrt{b}-a}{2\sqrt{b}} \right) \tag{4.16}$$

and an expression in terms of the Jacobi polynomials $P_n^{(\alpha,\beta)}$ (see [2]):

$$\mathcal{C}_n(a,b) = \frac{\pi}{n(2\sqrt{b})^n} \frac{1}{(a+\sqrt{b})^{n+1}} P_{n-1}^{(n+1,-n-1)}\left(\frac{a}{\sqrt{b}}\right). \tag{4.17}$$

5. The motivating example

As mentioned in the Introduction, the sequence that lead the authors to the present work results from the evaluation of the quartic integral

$$N_{0,4}(a;n) = \int_0^{\infty} \frac{dx}{(x^4+2ax^2+1)^{n+1}}. \tag{5.1}$$

The main result of [7] is that the expression

$$P_n(a) = \frac{1}{\pi} 2^{n+3/2} (a+1)^{n+1/2} N_{0,4}(a;n) \tag{5.2}$$

is a *polynomial* in a, of degree n, with the coefficient of a^i given by

$$d_i(n) = \sum_{k=i}^{n} 2^{k-2n} \binom{2n-2k}{n-k} \binom{n+k}{k} \binom{k}{i}. \tag{5.3}$$

Properties of these coefficients are studied in [16]. In particular, for fixed n, the sequence $(d_i(n))_i$ was shown to be unimodal in [1, 5, 8]. Its log-concavity was established in [13] and its 2-log-concavity appeared in [10]. The question about the infinite log-concavity of $\{d_i(n)\}_i$ remains open. The next statement follows from Proposition 1:

Proposition 2. *For fixed $r \in \mathbb{N}$, the sequence $\{P_n(r)\}_n$ is infinitely log-convex.*

Proof. Proposition 1 and the integral representation

$$P_n(r) = \frac{2^{3/2}\sqrt{r+1}}{\pi} \int_0^\infty \left(\frac{2(r+1)}{x^4+2rx^2+1} \right)^n d\mu(x) \tag{5.4}$$

with $d\mu(x) = \dfrac{dx}{x^4+2rx^2+1}$, yield the result. □

6. Chromatic polynomials

This last section discusses properties of chromatic polynomials of graphs. Recall that given an undirected graph G and x distinct colors, the number of proper colorings (adjacent vertices having distinct colors) is a polynomial in x, called the chromatic polynomial of G and denoted by $\kappa_G(x)$.

Examples of chromatic polynomials include

- If G is a graph with n vertices and no edges, then $\kappa_G(x) = x^n$;
- If G is a tree with n vertices, then $\kappa_G(x) = x(x-1)^{n-1}$;
- If G is the complete graph with n vertices, then

$$\kappa_G(x) = x(x-1)\cdots(x-n+1).$$

In these examples, the chromatic polynomials have only real roots. The log-concavity of the coefficients follows from a work of P. Bränden [9].

Other examples of chromatic polynomials include

- For a cycle G with n vertices, $\kappa_G(x) = (x-1)^n + (-1)^n(x-1)$;
- If G is the bipartite graph $K_{n,m}$, then

$$\kappa_G(x) = \sum_{j=0}^{m} S(m,j)(x)_j(x-j)^n,$$

  where $S(m,k)$ is the Stirling number of the second kind and $(x)_k = x(x-1)\cdots(x-k+1)$ is the falling factorial.
- If G is the cyclic ladder graph with $2n$ vertices, then

$$\kappa_G(x) = (x^2-3x+3)^n - (1-x)^{n+1} - (1-x)(3-x)^n + (x^2-3x+1). \tag{6.1}$$

- If G is the signed book graph $B(m,n)$, then

$$\kappa_G(x) = (x-1)^m x^{-n} \left((x-1)^m + (-1)^m\right)^n. \tag{6.2}$$

These examples, as well as many more from the long list given by Birkhoff and Lewis [4], have been tested to be infinitely log-concave.

J. Huh [12] proved:

Theorem 20. *The absolute values of the coefficients of a chromatic polynomial $\kappa_G(x)$ are log-concave.*

The authors will analyze chromatic polynomials by the methods presented in this paper. In the meantime, based on some experimental evidence, we invite the reader to:

Conjecture 21. *The absolute values of the coefficients of any chromatic polynomial are infinitely log-concave.*

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