STARLIKENESS OF JANOWSKI SPIRALLIKE FUNCTIONS IN THE UNIT DISK

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Abstract. The aim of this paper is to introduce and study new class of analytic functions which generalize the classes of λ-Spirallike Janowski functions, In particular. We gave the representation theorem, the right said of the covering theorem, starlikeness estimates and some properties related to the functions in the class $S^{\lambda}(T, H, F)$

1. **Introduction**

Let $\mathcal{W} = \{\vartheta \in \mathbb{C} : |\vartheta| < 1\}$ be the open unit disk and $\mathcal{M}$ denote the class of functions has a Taylor−Maclaurin's series representation $h(\vartheta) = \vartheta + \sum_{i=2}^{\infty} a_i \vartheta^i$. Suppose that $\mathcal{S}$ the class of functions in $\mathcal{M}$ and univalent in $\mathcal{W}$. The convolution or Hadamard product of two analytic functions $h, g \in \mathcal{M}$ where h is defined above and $g(\vartheta) = \vartheta + \sum_{i=2}^{\infty} b_i \vartheta^i$, is

$$(h * g)(\vartheta) = \vartheta + \sum_{i=2}^{\infty} a_i b_i \vartheta^i.$$

which are analytic in the open unit disk , and $\mathcal{S}$ denote the subclass of $\mathcal{M}$ consisting of all function which are univalent in $\mathcal{U}$.

For h and g be in $\mathcal{M}$, we say that the function h is *subordinate* to g in $\mathcal{W}$, and write $h \prec g$, if there exists a *Schwarz function* $u \in \Phi$, where

$$\Phi := \{u \in \mathcal{M}, \ u(0) = 0, |u(\vartheta)| < 1, \ \vartheta \in \mathcal{W}\}, \tag{1.1}$$

such that $h(\vartheta) = g(u(\vartheta))$, $\vartheta \in \mathcal{W}$. If g is univalent in $\mathcal{W}$, then

$$h(\vartheta) \prec g(\vartheta) \Leftrightarrow h(0) = g(0) \text{ and } h(\mathcal{W}) \subset g(\mathcal{W}), \ \vartheta \in \mathcal{W}.$$

Using the above principle of the subordination we define the well-known Carathéodory class $\mathcal{P}$ of functions p which are analytic in $\mathcal{W}$, which are normalized by

$$p(\vartheta) = 1 + \sum_{j=1}^{\infty} c_j \vartheta^j \text{ and } \Re\{p(\vartheta)\} > 0, \ \vartheta \in \mathcal{W}.$$

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In [7], Polatoğlu, et al. defined and studied a generalized class $\mathcal{P}[H,T,F]$ of Janowski functions. A function p in $\mathcal{P}$ is said to be in class $\mathcal{P}[H,T,F]$ if it satisfies the condition

$$p(\vartheta) \prec \frac{1+[(1-F)T+FH]\vartheta}{1+H\vartheta} \Leftrightarrow p(\vartheta) = \frac{1+[(1-F)T+FH]u(\vartheta)}{1+Hu(\vartheta)},\ u \in \Phi,$$

where $-1 \leq B < T \leq 1$ and $0 \leq F < 1$.

Definition 1.1. *A function $f \in \mathcal{M}$ is said to belongs to the class $\mathcal{S}^{\lambda}(T,H,F)$, with $-1 \leq H < T \leq 1$ and $0 \leq F < 1$, if*

$$\frac{\frac{e^{i\lambda}\vartheta h'(\vartheta)}{h(\vartheta)} - i\sin(\lambda)}{\cos(\lambda)} \in \mathcal{P}[H,T,F].$$

Remark 1.2. *Using the principle of the subordination we can easily obtain that the equivalent condition for a function h belonging to the class $\mathcal{S}^{\lambda}(T,H,F)$, is*

$$\left|\frac{\vartheta h'(\vartheta)}{h(\vartheta)} - 1\right| < \left|[\gamma - H\frac{\vartheta h'(\vartheta)}{h(\vartheta)}\right|,\ \vartheta \in \mathcal{W}.$$

where

$$\gamma = e^{-i\lambda}\{[(1-F)T+FH]\cos(\lambda) + iH\sin(\lambda)\}. \tag{1.2}$$

For special cases for the parameters λ, T, H and F the class $\mathcal{S}^{\lambda}(T,H,0) = \mathcal{S}^{\lambda}(T,H)$ Motivated by Polatoğlu, et al. [8], $\mathcal{S}^{0}(T,H,E) = \mathcal{S}(T,H,F)$ Motivated by Polatoğlu, et al. [7], $\mathcal{S}^{0}(T,H,0) = \mathcal{S}(T,H)$ these class reduce to well-known class defined by Janowski [6], $\mathcal{S}^{0}(1-2\eta,-1,0) = \mathcal{S}(\eta)$ the well-known class of starlike function of order alpha by Robertson [2], $\mathcal{S}^{0}(1,-1,0) = \mathcal{S}^{*}$ the class introduced by Nevanlinna [4], etc.

Lemma 1.3. [1] *Let $u(\vartheta)$ be regular in the unit disk $\mathcal{W}$ with $u(0) = 0$. Then if $|u(\vartheta)|$ obtains its maximum value on the circle $|\vartheta| = r$ at the point ϑ_1, one has $\vartheta_1 u'(\vartheta_1) = ku(\vartheta)$, for some $k \geq 1$.*

Lemma 1.4. [5, Lemma 2.3] *Let $p \in \mathcal{P}[T,H,F]$, then*

$$\frac{1-[(1-F)T+FH]r}{1-Hr} \leq |p(\vartheta)| \leq \frac{1+[(1-F)T+FH]r}{1+Hr},\ |\vartheta| \leq r < 1.$$

Lemma 1.5. [7] *Let $p \in \mathcal{P}[T,H,F]$, then the set of the values of p is in the closed disc with center at $C(r)$ and having the radius $\rho(r)$, where*

$$\begin{cases} C(r) = \left(\frac{1-H[(1-F)T+FH]r^2}{1-H^2r^2}, 0\right), \rho(r) = \frac{(1-F)(T-H)r}{1-H^2r^2} & \text{if} \quad H \neq 0, \\ C(r) = (1,0), \rho(r) = (1-F)|T|r & \text{if} \quad H = 0. \end{cases}$$

2. Main results

Theorem 2.1.

$$h \in \mathcal{S}^{\lambda}(T,H,F) \Leftrightarrow \left(\frac{\vartheta h'(\vartheta)}{h(\vartheta)}-1\right) \prec \begin{cases} \frac{e^{-i\lambda}(1-F)(T-H)\cos(\lambda)\vartheta}{1+H\vartheta}, & H \neq 0, \\ e^{-i\lambda}(1-F)T\cos(\lambda)\vartheta, & H = 0. \end{cases} \tag{2.1}$$

Proof. Let $h \in \mathcal{S}^{\lambda}(T,H,F)$, we define the functions $u(\vartheta)$ by

$$\frac{h(\vartheta)}{\vartheta} = \begin{cases} (1+Hu(\vartheta))^{\frac{e^{-i\lambda}(1-F)(T-H)\cos(\lambda)}{H}}, & H \neq 0, \\ e^{(1-F)T\cos(\lambda)e^{-i\lambda}u(\vartheta)}, & H = 0. \end{cases} \tag{2.2}$$

For $\vartheta = 0$, we note that

$$(1+Hu(\vartheta))^{\frac{e^{-i\lambda}(1-F)(T-H)\cos(\lambda)}{H}} = 1 = e^{(1-F)T\cos(\lambda)e^{-i\lambda}u(\vartheta)}.$$

That mean is analytic and $u(0) = 0$, if we take the logarithmic derivative from 2.2, we get

$$\left(\frac{\vartheta h'(\vartheta)}{h(\vartheta)}-1\right) = \begin{cases} \frac{e^{-i\lambda}(1-F)(T-H)\cos(\lambda)\vartheta u'(\vartheta)}{1+Hu(\vartheta)}, & H \neq 0, \\ e^{-i\lambda}(1-F)T\cos(\lambda)\vartheta u'(\vartheta), & H = 0. \end{cases}$$

We can easily conclude that this subordination is equivalent to $|u(\vartheta)| < 1$ for all $z \in \mathcal{W}$. On the contrary let's assume that there exists $z_1 \in \mathcal{W}$, such that $|u(\vartheta)|$ attains its maximum value on the circle $|\vartheta| = r$, that is $|u(\vartheta_1)| = 1$. Then when the conditions $\vartheta_1 u'(z_1) = ku(\vartheta_1), k \geq 1$ are satisfied for such $\vartheta_1 \in \mathcal{W}$, by using Lemma 1.3 we obtain,

$$\left(\frac{\vartheta_1 h'(\vartheta_1)}{h(\vartheta_1)}-1\right) = \begin{cases} \frac{e^{-i\lambda}(1-F)(T-H)\cos(\lambda)ku(\vartheta_1)}{1+Hu(\vartheta_1)} = G_1(u(\vartheta_1)) \notin G_1(\mathcal{W}), & H \neq 0, \\ e^{-i\lambda}(1-F)T\cos(\lambda)ku(\vartheta_1) = G_2(u(\vartheta_1)) \notin G_2(\mathcal{W}), & H = 0, \end{cases}$$

which contradicts 2.1 therefore the assumption is wrong , i.e,
$|u(\vartheta)| < 1$ for all $\vartheta \in \mathcal{W}$. This shows that

$$h \in \mathcal{S}^{\lambda}(T,H,F) \Rightarrow \left(\frac{\vartheta h'(\vartheta)}{h(\vartheta)}-1\right) \prec \begin{cases} \frac{e^{-i\lambda}(1-F)(T-H)\cos(\lambda)\vartheta}{1+H\vartheta}, & H \neq 0, \\ e^{-i\lambda}(1-\alpha)A\cos(\lambda)z, & B = 0. \end{cases}$$

Conversely,

$$\left(\frac{\vartheta h'(\vartheta)}{h(\vartheta)}-1\right) \prec \begin{cases} \frac{e^{-i\lambda}(1-F)(T-H)\cos(\lambda)\vartheta}{1+H\vartheta}, & H \neq 0, \\ e^{-i\lambda}(1-F)T\cos(\lambda)\vartheta, & H = 0. \end{cases} \Rightarrow$$

$$\frac{e^{-i\lambda}\frac{\vartheta h'(\vartheta)}{h(\vartheta)} - i\sin(\lambda)}{\cos(\lambda)} = \begin{cases} \frac{1+[(1-F)T+FH]u(\vartheta)}{1+Hu(\vartheta)}, & H \neq 0, \\ 1+[(1-F)T+FH]u(\vartheta), & H = 0. \end{cases}$$

This shows that $h(\vartheta) \in \mathcal{S}^{\lambda}(T,H,F)$. □

Remark 2.2. *In* [7] *and* [8], *a similar technique using Jack's lemma was used to investigate Janowski starlikeness.*

Theorem 2.3. *If $h \in \mathcal{S}^{\lambda}(T,H,F)$, then $|h(\vartheta)| \leq$*

$$\begin{cases} \int_0^r \left[\cos(\lambda)\dfrac{1+[(1-F)T+FH]\sigma}{1+H\sigma}+|\sin(\lambda)|\right](1+H\sigma)^{\frac{e^{-i\lambda}(1-F)(T-H)\cos(\lambda)}{H}}d\sigma, & H\neq 0, \\ \int_0^r [\cos(\lambda)[1+(1-F)T\sigma]+|\sin(\lambda)|]\exp\left[(1-F)T\cos(\lambda)(\cos(\lambda)+\sin(\lambda))\sigma\right]d\sigma, & H=0, \end{cases}$$

where $|\vartheta| \leq r < 1$.

Proof. Integrating the function h' along the close segment connecting the origin with an arbitrary $\vartheta \in \mathcal{W}$, and observing that a point on this segment is of the form $\zeta = \sigma e^{i\theta}$, with $\sigma \in [0,r]$, where $\theta = \arg\vartheta$ and $r = |\vartheta|$, we get

$$h(\vartheta) = \int_0^{\vartheta} h'(\zeta)d\zeta,\ \vartheta = re^{i\theta},$$

hence

$$|h(\vartheta)| = \left|\int_0^r h'\left(\sigma e^{i\theta}\right)e^{i\theta}d\sigma\right| \leq \int_0^r \left|h'\left(\sigma e^{i\theta}\right)e^{i\theta}\right|d\sigma.$$

For an arbitrary function $h \in \mathcal{S}^{\lambda}(T,H,F)$, we have

$$\frac{1}{\cos(\lambda)}\left[\frac{e^{i\lambda}\vartheta h'(\vartheta)}{h(\vartheta)} - i\sin(\lambda)\right] = p(\vartheta), \qquad p \in \mathcal{P}[T,H,F].$$

We need to study the following cases:

(i) If $H \neq 0$, then there exists a function $u \in \Phi$, such that by (2.2) we get $h(\vartheta) = \vartheta(1+Hu(\vartheta))^{\frac{e^{-i\lambda}(1-F)(T-H)\cos(\lambda)}{H}}$, and

$$h'(\vartheta) = \frac{h(\vartheta)}{\vartheta}e^{-i\lambda}[\cos(\lambda)p(\vartheta)+i\sin(\lambda)],$$

therefore

$$|h'(\vartheta)| \leq \left[\cos(\lambda)\frac{1+[(1-F)T+FH]r}{1+Hr}+|\sin(\lambda)|\right]|1+Hu(\vartheta)|^{\frac{e^{-i\lambda}(1-F)(T-H)\cos(\lambda)}{H}}, \tag{2.3}$$

for $|\vartheta| \leq r < 1$. Since $u \in \Phi$, we have

$$|1+Hu(\vartheta)| \leq 1+|H|r,\ |\vartheta| \leq r < 1.$$

Case 1. If $H > 0$, using the fact that $-1 \leq H < T \leq 1$ and $0 \leq F < 1$, we have

$$|1+Hu(\vartheta)|^{\frac{e^{-i\lambda}(1-F)(T-H)\cos(\lambda)}{H}} \leq (1+|H|r)^{\frac{e^{-i\lambda}(1-F)(T-H)\cos(\lambda)}{H}},\ |\vartheta| \leq r < 1,$$

and from (2.3) for $|\vartheta| \leq r < 1$, we obtain

$$|h'(\vartheta)| \leq \left[\cos(\lambda)\frac{1+[(1-F)t+FH]r}{1+Hr}+|\sin(\lambda)|\right](1+|h|r)^{\frac{e^{-i\lambda}(1-F)(T-H)\cos(\lambda)}{H}}. \tag{2.4}$$

Case 2. If $H < 0$, from the fact that $-1 \leq H < T \leq 1$ and $0 \leq F < 1$, we have

$$(1-|H|r)^{\frac{e^{-i\lambda}(1-F)(T-H)\cos(\lambda)}{H}} \geq |1+Hu(\vartheta)|^{\frac{e^{-i\lambda}(1-F)(T-H)\cos(\lambda)}{H}},\ |\vartheta| \leq r < 1,$$

and from (2.3) for $|\vartheta| \leq r < 1$, we obtain

$$\left[\cos(\lambda)\frac{1+[(1-F)T+FH]r}{1+Hr} + |\sin(\lambda)|\right](1-|H|r)^{\frac{e^{-i\lambda}(1-F)(T-H)\cos(\lambda)}{H}} \geq |h'(\vartheta)|. \tag{2.5}$$

Now, combining the inequalities (2.4) and (2.5), we finally conclude that

$$|h'(\vartheta)| \leq \left[\cos(\lambda)\frac{1+[(1-F)T+FH]r}{1+Hr} + |\sin(\lambda)|\right](1+Hr)^{\frac{(1-F)(T-H)}{H}},\ |\vartheta| \leq r < 1, \tag{2.6}$$

then

$$|h(\vartheta)| \leq \int_0^r \left|h'\left(\sigma e^{i\theta}\right)e^{i\theta}\right| d\sigma \leq$$

$$\int_0^r \left[\cos(\lambda)\frac{1+[(1-F)T+FH]\sigma}{1+H\sigma} + |\sin(\lambda)|\right](1+H\sigma)^{\frac{e^{-i\lambda}(1-F)(T-H)\cos(\lambda)}{H}} d\sigma,$$

that is

$$|h(\vartheta)| \leq \int_0^r \left[\cos(\lambda)\frac{1+[(1-F)T+FH]\sigma}{1+H\sigma} + |\sin(\lambda)|\right](1+H\sigma)^{\frac{e^{-i\lambda}(1-F)(T-h)\cos(\lambda)}{H}} d\sigma,\ |\vartheta| \leq r < 1.$$

(ii) If $B = 0$, there exists a function $u \in \Phi$, such that $h(\vartheta) = \vartheta e^{(1-F)T\cos(\lambda)e^{-i\lambda}u(\vartheta)}$, and therefore

$$|h'(\vartheta)| \leq \{\cos(\lambda)[1+(1-F)Tr] + |\sin(\lambda)|\}\left|\exp\left[(1-F)T\cos(\lambda)e^{-i\lambda}u(\vartheta)\right]\right|,\ |\vartheta| \leq r < 1. \tag{2.7}$$

Since $\left|\exp\left[(1-F)T\cos(\lambda)e^{-i\lambda}u(\vartheta)\right]\right| = \exp\left[((1-F)T\cos(\lambda)\Re\{e^{-i\lambda}u(\vartheta)\}\right]$, $\vartheta \in \mathcal{W}$, using a similar computation as in the previous case, we deduce

$$\left|\exp\left[(1-F)T\cos(\lambda)e^{-i\lambda}u(\vartheta)\right]\right| \leq \exp\left[(1-F)T\cos(\lambda)(\cos(\lambda)+\sin(\lambda))r\right],\ |\vartheta| \leq r < 1.$$

Thus, (2.7) yields

$$|h'(\vartheta)| \leq [\cos(\lambda)[1+(1-F)Tr] + |\sin(\lambda)|]\exp\left[(1-F)T\cos(\lambda)(\cos(\lambda)+\sin(\lambda))r\right],\ r < 1, \tag{2.8}$$

and hence

$$|h(\vartheta)| \leq \int_0^r \left|h'\left(\sigma e^{i\theta}\right)e^{i\theta}\right| d\sigma \leq$$

$$\int_0^r [\cos(\lambda)[1+(1-F)T\sigma] + |\sin(\lambda)|]\exp\left[(1-F)T\cos(\lambda)(\cos(\lambda)+\sin(\lambda))\sigma\right]]d\sigma,$$

that is

$$|h(\vartheta)| \leq \int_0^r [\cos(\lambda)[1+(1-F)T\sigma] + |\sin(\lambda)|]\exp\left[(1-F)T\cos(\lambda)(\cos(\lambda)+\sin(\lambda))\sigma\right]d\sigma,\ r < 1.$$

□

Theorem 2.4. *The radius of starlikeness of the class $h \in \mathcal{S}^{\lambda}(T,H,F)$ is*

$$r = \begin{cases} \dfrac{2}{(1-F)(T-H)\cos(\lambda)+\sqrt{(1-F)^2(T-H)^2\cos^2(\lambda)+4\left\{[(1-F)T+FH]H\cos^2(\lambda)+H^2\sin^2(\lambda)\right\}}}, & \text{if} \quad H \neq 0, \\ \dfrac{1}{(1-F)T\cos(\lambda)}, & \text{if} \quad H = 0. \end{cases}$$

This radius is sharp because the extremal function is

$$h(\vartheta) = \begin{cases} \vartheta(1+H\vartheta)^{\frac{e^{-i\lambda}(1-F)(T-H)\cos(\lambda)}{H}}, & H \neq 0, \\ \vartheta e^{(1-F)T\cos(\lambda)e^{-i\lambda}\vartheta}, & H = 0. \end{cases}.$$

Proof. Since

$$\frac{\dfrac{e^{i\lambda}\vartheta h'(\vartheta)}{h(\vartheta)} - i\sin(\lambda)}{\cos(\lambda)} = p(\vartheta), \qquad p \in \mathcal{P}[T,H,F], \tag{2.9}$$

using Lemma 1.5, that is

$$\left| p(\vartheta) - \frac{1-H[(1-F)T+FH]r^2}{1-H^2r^2} \right| \leq \frac{(1-F)(T-H)r}{1-H^2r^2}. \tag{2.10}$$

Using (2.9) in (2.10) and after straightforward calculations we get

$$\left.\begin{matrix} \frac{1-(1-F)(T-H)\cos(\lambda)r-\left\{[(1-F)T+FH]H\cos^2(\lambda)+H^2\sin^2(\lambda)\right\}r^2}{1-H^2r^2}, & \text{if} \quad H \neq 0, \\ 1-(1-F)T\cos(\lambda)r, & \text{if} \quad H = 0 \end{matrix}\right\} \leq \Re\left\{\vartheta\frac{h'(\vartheta)}{h(\vartheta)}\right\} \leq$$

$$\begin{cases} \frac{1+(1-F)(T-H)\cos(\lambda)r-\left\{[(1-F)T+FH]H\cos^2(\lambda)+H^2\sin^2(\lambda)\right\}r^2}{1-H^2r^2}, & \text{if} \quad H \neq 0, \\ 1+(1-F)T\cos(\lambda)r, & \text{if} \quad H = 0, \end{cases}$$

where $|\vartheta| \leq r < 1$. The above inequalities shows that this theorem is true. □

Remark 2.5.

- *For $T = -H = 1, \lambda = F = 0$, we obtain $r = 1$.*
- *For $T = -H = 1, F = 0$, we obtain $r = \frac{1}{\cos(\lambda)+|\sin(\lambda)|}$.*

Corollary 2.6. *If $h \in \mathcal{S}^{\lambda}(T,H,F)$, then*

$$\left.\begin{matrix} \frac{(1-[(1-F)T+FH]r)\cos(\lambda)-(1-Hr)|\sin(\lambda)|}{1-Hr}, & \text{if} \quad H \neq 0, \\ (1-(1-F)Tr)\cos(\lambda)-|\sin(\lambda)|, & \text{if} \quad H = 0 \end{matrix}\right\} \leq \left|\frac{\vartheta h'(\vartheta)}{h(\vartheta)}\right| \leq$$

$$\begin{cases} \frac{(1+[(1-F)T+FH]r)\cos(\lambda)+(1+Hr)|\sin(\lambda)|}{1+Hr}, & \text{if} \quad H \neq 0, \\ (1+(1-F)Tr)\cos(\lambda)+|\sin(\lambda)|, & \text{if} \quad H = 0, \end{cases}$$

where $|\vartheta| \leq r < 1$.

Proof. For an arbitrary function $h \in \mathcal{S}^{\lambda}(T, H, F)$, we have

$$\frac{1}{\cos(\lambda)}\left[\frac{e^{i\lambda}\vartheta h'(\vartheta)}{h(\vartheta)} - i\sin(\lambda)\right] = p(\vartheta), \qquad p \in \mathcal{P}[T, H, F].$$

Using Lemma 1.4 and after the straightforward calculations we get the result. □

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