POWER COMPOSITIONS AND SEMI-PELL COMPOSITIONS

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ABSTRACT. In analogy with the semi-Fibonacci partitions studied recently by Andrews, we define semi-m-Pell compositions. We find that these are in bijection with certain weakly unimodal m-ary compositions. We give generating functions, bijective proofs, and a number of unexpected congruences for these objects. In the special case of $m = 2$, we have a new combinatorial interpretation of the semi-Pell sequence and connections to other objects.

1. INTRODUCTION

A composition of a positive integer n is an ordered partition of n, that is, any sequence of positive integers $(n_1, \ldots, n_k)$ such that $n_1 + \ldots + n_k = n$. Compositions of n will be represented as vectors with positive-integer entries.

A recent paper of Andrews defined *semi-Fibonacci partitions* [2]. Motivated by the philosophical position that where there is an interesting *partition* object there is often an interesting *composition* object, we study the set $SP(n, m)$ of semi-m-Pell compositions, defined as follows:

Definition. $SP(n, m) = \{(n)\},\ n = 1, 2, \ldots, m$. If $n > m$ and n is a multiple of m, then

$$SP(n, m) = \{m\lambda \mid \lambda \in SP(n/m, m)\}.$$

If n is not a multiple of m, that is, $n \equiv r \pmod{m}$, $1 \le r \le m - 1$, then $SP(n, m)$ arises from two sources: first, compositions obtained by inserting r at the beginning or at the end of each composition in $SP(n - r, m)$, and second, compositions obtained by adding m to the single part of each composition $\lambda \in SP(n - m, m)$ which is congruent to r (mod m). (Note that λ contains exactly one part which is congruent to r modulo m, see Lemma 1 below). If n is a multiple of m, then every semi-m-Pell composition of n arises from multiplying the parts in all semi-m-Pell compositions of $\frac{n}{m}$ by m.

As an illustration we have the following sets for small n:
$SP(1, 3) = \{(1)\}$,
$SP(2, 3) = \{(2)\}$,
$SP(3, 3) = \{(3)\}$,
$SP(4, 3) = \{(1, 3), (3, 1), (4)\}$,
$SP(5, 3) = \{(2, 3), (3, 2), (5)\}$,

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$SP(6,3) = \{(6)\}$,
$SP(7,3) = \{(1,6),(6,1),(4,3),(3,4),(7)\}$,
$SP(8,3) = \{(2,6),(6,2),(5,3),(3,5),(8)\}$,
$SP(9,3) = \{(9)\}$,
$SP(10,3) = \{(1,9),(9,1),(4,6),(6,4),(7,3),(3,7),(10)\}$.

Thus if we define $sp(n,m) = |SP(n,m)|$, then the following recurrence relation holds:

$$sp(n,m) = 0 \text{ if } n \leq 0, \text{ and } sp(n,m) = 1 \text{ for } 1 \leq n \leq m-1.$$

Then for $n \geq m$,

$$sp(n,m) = \begin{cases} sp(n/m,m) & \text{if } n \equiv 0 \pmod{m}, \\ 2\,sp(n-r,m) + sp(n-m,m) & \text{if } n \equiv r \pmod{m}, 0 < r < m. \end{cases} \tag{1.1}$$

The semi-Pell sequence $\{sp(n)\}_{n>0}$ occurs as sequence number A129095 in the Online Encyclopedia of Integer Sequences [8], where

$$\begin{cases} sp(n/2) & \text{if } n \text{ is even}, \\ 2\,sp(n-1) + sp(n-2) & \text{if } n \text{ is odd}, n > 1. \end{cases} \tag{1.2}$$

However, there seems to be no connection of the sequence with compositions until now. The companion sequence A129096 records the fact that the bisection of the semi-Pell sequence $sp(2n-1)$, $n > 0$ is monotonically increasing: $sp(2n+3) = 2sp(2n+2) + sp(2n+1) > sp(2n+1)$ for all $n \geq 0$.

In Section 2 we relate semi-m-Pell compositions to a restricted class of weakly unimodal compositions. We also give an alternative characterization of semi-m-Pell compositions in Section 3. Then in Section 4 we prove some congruences satisfied by the enumeration function of these compositions. The most surprising of these is certainly Theorem 5, which states a congruence mod 3 for semi-m-Pell compositions in certain square-modulus arithmetic progressions. In Section 5 we briefly look at the special case $m = 2$, the original motivation for this project, which are enumerated by the semi-2-Pell sequence. .

2. The Semi-m-Pell Compositions

Lemma 1. Let $\lambda \in SP(n,m)$.

(i) If $m \mid n$, then every part of λ is a multiple of m.

(ii) If $n \equiv r \pmod{m}$, $1 \leq r < m$, then λ contains exactly one part $\equiv r \pmod{m}$.

Proof. If $m \mid n$, the parts of a composition in $SP(n,m)$ are divisible by m by construction.

For induction note that $SP(r,m) = \{(r)\}$, $r = 1,\ldots,m-1$; so the assertion holds trivially. Assume that the assertion holds for the compositions of all integers $< n$ and consider $\lambda \in SP(n,m)$ with $1 \leq r < m$. Then λ may be obtained by inserting r at the beginning or end of a composition $\alpha \in SP(n-r,m)$. Since α consists of multiples of m (as $m|(n-r)$), λ contains exactly one part $\equiv r \pmod{m}$. Alternatively

λ is obtained by adding m to the single part of a composition $\beta \in SP(n-m,m)$ which is $\equiv r \pmod{m}$. Indeed β contains exactly one such part by the inductive hypothesis. Hence the assertion is proved. □

We will associate the set of semi-m-Pell compositions with a class of restricted unimodal compositions into powers of $m > 1$.

A weakly unimodal composition (or stack) is defined to be any composition of the form $(a_1, a_2, \ldots, a_r, c, b_s, \ldots, b_1)$ such that

$$1 \leq a_1 \leq a_2 \leq \cdots \leq a_r \leq c > b_s \geq \cdots \geq b_2 \geq b_1.$$

The set of the a_i or the b_j may be empty. The study of these compositions was pioneered by Auluck [5] and Wright [9] and continues to instigate research (see, for example [2, 3, 6]). The "concave" compositions studied by Andrews in [3] are weakly unimodal compositions with a unique largest part.

Let $oc(n,m)$ denote the number of weakly unimodal m-power compositions of n in which every part size occurs together or in one "place", with multiplicity not divisible by m (see also Munagi-Sellers [7]). Thus for example, the following are some objects enumerated by $oc(92,3)$: $(27^2, 9^2, 3^2, 1^{14}), (1^8, 3^{13}, 9^2, 27)$ and $(3^{14}, 9, 27, 1^{14})$. However, the following weakly unimodal binary compositions of 92 do not belong to $oc(92,3)$: $(1^8, 3^{13}, 9^2, 4^6, 3), (1^3, 3^{13}, 27, 9^2, 1^5)$.

Theorem 1. *For integers $n \geq 1, m > 1$,*

$$sp(n,m) = oc(n,m). \tag{2.1}$$

Their common generating function is

$$\sum_{i=0}^{\infty} \frac{\sum_{r=1}^{m-1} x^{m^i r}}{1 - x^{m^{i+1}}} \prod_{t=0}^{i-1} \left(1 + \frac{2\sum_{r=1}^{m-1} x^{m^t r}}{1 - x^{m^{t+1}}}\right). \tag{2.2}$$

Proof. We prove first the generating function claim, and then give a bijection.

First Proof (generating functions). Let $Q_m(x) = \sum_{n \geq 0} sp(n,m) x^n$. Then

$$\begin{aligned}
Q_m(x) &= \sum_{n\geq 0} sp(n,m)x^{nm} + \sum_{r=1}^{m-1}\sum_{n\geq 0} sp(nm+r,m)x^{nm+r} \\
&= \sum_{n\geq 0} sp(n,m)x^{nm} + \sum_{r=1}^{m-1}\sum_{n\geq 1} sp(nm+r,m)x^{nm+r} + \sum_{r=1}^{m-1} sp(r,m)x^r \\
&= Q_m(x^m) + \sum_{r=1}^{m-1}\sum_{n\geq 1} (2sp(nm,m) + sp(nm+r-m,m))x^{nm+r} + \sum_{r=1}^{m-1} x^r \\
&= Q_m(x^m) + 2\sum_{r=1}^{m-1} x^r \sum_{n\geq 1} sp(nm,m)x^{nm} + \sum_{r=1}^{m-1}\sum_{n\geq 0} sp(nm+r,m)x^{nm+r+m} + \sum_{r=1}^{m-1} x^r \\
&= Q_m(x^m) + 2Q_m(x^m)\sum_{r=1}^{m-1} x^r + \sum_{r=1}^{m-1} x^r + \sum_{r=1}^{m-1}\sum_{n\geq 0} sp(nm+r,m)x^{nm+r+m}.
\end{aligned}$$

Eliminating the last sum by means of the first equality, we obtain

$$Q_m(x) = Q_m(x^m) + 2(Q_m(x^m)\sum_{r=1}^{m-1} x^r + \sum_{r=1}^{m-1} x^r + x^m(Q_m(x) - Q_m(x^m)),$$

$$(1-x^m)Q_m(x) = Q_m(x^m) + 2Q_m(x^m)\sum_{r=1}^{m-1} x^r + \sum_{r=1}^{m-1} x^r - x^m Q_m(x^m),$$

which gives the functional equation

$$Q_m(x) = \frac{\sum_{r=1}^{m-1} x^r}{1-x^m} + \frac{1+2\sum_{r=1}^{m-1} x^r - x^m}{1-x^m} Q_m(x^m).$$

The functional equation can be iterated. In the combinatorial limit we obtain

$$Q_m(x) = \sum_{i=0}^{\infty} \frac{\sum_{r=1}^{m-1} x^{m^i r}}{1-x^{m^{i+1}}} \prod_{t=0}^{i-1} \frac{1+2\sum_{r=1}^{m-1} x^{m^t r} - x^{m^{t+1}}}{1-x^{m^{t+1}}}.$$

Finally, we see that

$$Q_m(x) = \sum_{i=0}^{\infty} \frac{\sum_{r=1}^{m-1} x^{m^i r}}{1-x^{m^{i+1}}} \prod_{t=0}^{i-1} \left(1 + \frac{2\sum_{r=1}^{m-1} x^{m^t r}}{1-x^{m^{t+1}}}\right) = \sum_{n=0}^{\infty} oc(n,m)x^n$$

is also the generating function for $oc(m,n)$, since the i term counts compositions in which the largest part is m^i, and there are two possible places for the parts of size m^t for $t < i$, if any appear at all, since these must appear with powers unimodally increasing and then decreasing. This completes the first proof.

Second Proof (bijection). Let the sets enumerated by $sp(n,m)$ and $oc(n,m)$ be denoted by $SP(n,m)$ and $OC(n,m)$ respectively.

Each part t of $C \in SP(n,m)$ can be expressed as $t = m^i \cdot h$, $i \geq 0$, where $m \nmid h$. Now transform t as follows:

$$t = m^i \cdot h \longmapsto m^i, m^i, \ldots, m^i \ (h \text{ times}).$$

Note that the case $i = 0$ may arise only as a first or last part of C. This gives a unique member of $OC(n,m)$ provided that we retain the clusters of the m^i corresponding to each t, in consecutive positions, and maintain the order of the parts of the resulting m-power composition.

To reverse the map we write each $\beta \in OC(n,m)$ in the one-place exponent notation, to get $\beta = (\beta_1^{u_1}, \ldots, \beta_s^{u_s})$ with the $m \nmid u_i$, and containing at most one instance of a 1-cluster which may be $\beta_1^{u_1}$ or $\beta_s^{u_s}$. Since each β_i has the form m^{j_i}, $j_i \geq 0$, we apply the transformation:

$$\beta_i^{u_i} = (m^{j_i})^{u_i} \longmapsto m^{j_i} \cdot u_i.$$

This gives a unique composition in $SP(n,m)$ provided that the resulting parts retain their relative positions. Indeed the image may contain at most one part $\equiv r \pmod{m}$ which occurs precisely when $j_i = 0$.

We illustrate the bijection with $(14, 3, 18, 27) \in SP(62, 3)$:

$$(14, 3, 18, 27) = (3^0 \cdot 14, 3^1 \cdot 1, 3^2 \cdot 2, 3^3 \cdot 1) \mapsto (1^{14}, 3, 9^2, 27) \in OC(62, 3).$$

□

We provide a full example with $n = 13, m = 3$, where $sp(13, 3) = 13 = oc(13, 3)$. The following members of the respective sets correspond 1-to-1 under the bijective proof of Theorem 1:

$SP(13, 3)$: $(1, 3, 9), (3, 9, 1), (1, 9, 3), (9, 3, 1), (1, 12), (12, 1), (4, 9), (9, 4), (7, 6), (6, 7), (10, 3), (3, 10), (13)$.

$OC(13, 3)$: $(1, 3, 9), (3, 9, 1), (1, 9, 3), (9, 3, 1), (1, 3^4), (3^4, 1), (1^4, 9), (9, 1^4), (1^7, 3^2), (3^2, 1^7), (1^10, 3), (3, 1^{10}), (1^{13})$.

Some coefficients in the expansion of $Q_m(x)$ are displayed in Table 1.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$m = 2$	1	1	3	1	5	3	11	1	13	5	23	3	29	11	51
$m = 3$	1	1	1	3	3	1	5	5	1	7	7	3	13	13	3
$m = 4$	1	1	1	1	3	3	3	1	5	5	5	1	7	7	7
$m = 5$	1	1	1	1	1	3	3	3	3	1	5	5	5	5	1
$m = 6$	1	1	1	1	1	1	3	3	3	3	3	1	5	5	5

TABLE 1. Values of $sp(n, m)$, for $2 \le m \le 6$, $1 \le n \le 15$

3. STRUCTURAL PROPERTIES OF SEMI-m-PELL COMPOSITIONS

Following [1] we define the *max m-power* of an integer N as the largest power of m that divides N (not just the exponent of the power). Thus using the notation $x_m(N)$, we find that $N = u \cdot m^s$, $s \ge 0$, where $m \nmid u$ and $x_m(N) = m^s$. So $x_m(N) > 0$ for all N. For example, $x_2(50) = 2$ and $x_5(216) = 1$.

Define three (reversible) operations on a composition $C = (c_1, \dots, c_k)$ with a fixed $m > 1$.

(i) If the first or last part of C is less than m, delete it: if $c_1 < m$ or $c_k < m$, then $\tau_1(C) = (c_2, \dots, c_k)$ or $\tau_1(C) = (c_1, \dots, c_{k-1})$ respectively;

(ii) If $m \nmid c_t > m$, $1 \le t \le k$, then $\tau_2(c) = (c_1, \dots, c_{t-1}, c_t - m, c_{t+1}, \dots, c_k)$;

(iii) If C consists of multiples of m, divide every part by m: $\tau_3(C) = (c_1/m, \dots, c_k/m)$.

These operations are consistent with the recursive construction of the set $SP(n, m)$, where τ_3^{-1}, τ_1^{-1} and τ_2^{-1} correspond, respectively, to the three quantities in the recurrence (1.1).

Lemma 2. Let $n > 0$, $m > 1$ be integers with $n \equiv r \pmod m$, $1 \le r < m$. If $C = (c_1, \dots, c_k) \in SP(n, m)$, then $c_1 \equiv r \pmod m$ or $c_k \equiv r \pmod m$.

Proof. If $k \leq 2$, the assertion is clear. So assume that $k > 2$ such that $c_i \equiv r \pmod{m}$ for a certain index $i \notin \{1, k\}$. Then we can apply τ_2 several times to obtain the composition $\beta = \tau_2^s(C)$, $s = \lfloor \frac{c_i}{m} \rfloor$, that contains r which is neither in the first nor last position. But this contradicts the recursive construction of β. Hence the assertion holds for all $C \in SP(n, m)$. □

Remark 1. If Lemma 2 is violated when $k > 2$, the sequence of max m-powers of the parts of C cannot be unimodal. If $m \nmid c_j$ with $c_j \notin \{c_1, c_k\}$, then $x_m(c_j) = 1$.

Lemma 3. Let $H(n, m)$ denote the set of compositions C of n such that the sequence of max m-powers of the parts of C are distinct and unimodal. If $C \in H(n, m)$ and $\tau_i(C) \neq \emptyset$, then $\tau_i(C) \in H(N, m)$, $i = 1, 2, 3$, for some N.

Proof. Let $C = (c_1, \ldots, c_k) \in H(n, m)$. If C contains a part r less than m, then $r = c_1$ or $r = c_k$ (by Lemma 2). So $\tau_1(C) \in H(n - r, m)$ since the max m-powers remain distinct and unimodal. If C contains a non-multiple of m, say $c_t > m$, then by Lemma 2, $t \in \{1, k\}$. Therefore $\tau_2(C)$, i.e., replacing c_t with $c_t - m$, preserves the unimodality of C. So $\tau_2(C) \in H(n - m, m)$. Lastly, since the parts of C have distinct max m-powers $\tau_3(C) = (c_1/m, \ldots, c_k/m)$ contains at most one non-multiple of m. Hence $\tau_3(C) \in H(n/m, m)$. □

We state an independent characterization of the semi-m-Pell compositions.

Theorem 2. *A composition C of n is a semi-m-Pell composition if and only if the sequence of max m-powers of the parts of C is distinct and unimodal.*

Proof. We show that $SP(n, m) = H(n, m)$. Let $C = (c_1, \ldots, c_k) \in SP(n, m)$ such that $C \notin H(n, m)$. Denote the properties,

P1: sequence of max m-powers of the parts of C is distinct.

P2: sequence of max m-powers of the parts of C is unimodal.

First assume that C satisfies P2 but not P1. So there are $c_i > c_j$ such that $x_m(c_i) = x_m(c_j)$, and let $c_i = u_i m^s$, $c_j = u_j m^s$ with $m \nmid u_i, u_j$. Observe that τ_1 deletes a part less than m, if it exists, from a member of $H(v, m)$. So we can use repeated applications of τ_2 to reduce a non-multiple modulo m, followed by τ_1. This is tantamount to simply deleting the non-multiple of m, say c_t, to obtain a member of $H(N, m)$, $N < v$, from Lemma 3. By thus successively deleting non-multiples from C, and applying τ_3^c, $c > 0$, we obtain a composition $E = (e_1, e_2, \ldots)$ with $e_i = v_i m^w > e_j = v_j m^w$, where $m \nmid v_i, v_j$ and $w \leq s$. Then apply τ_3^w to obtain a composition G with two non-multiples of m. Then by Lemma 1, $G \notin SP(n, m)$. Secondly assume that C satisfies P1 but not P2. Then by the proof of Lemma 2 and Remark 1, $\tau_2^u(C) \notin SP(N, m)$ for some u. Therefore $C \in SP(n, m) \implies C \in H(n, m)$.

Conversely let $C = (c_1, \ldots, c_k) \in H(n, m)$. If $C = (t)$, $1 \leq t \leq m$, then $C \in SP(t, m)$. If $m | c_i$ for all i, then $\tau_3(C) = (c_1/m, \ldots, c_k/m) \in H(n/m, m)$ contains at most one part $\not\equiv 0 \pmod{m}$, so $C \in SP(n, m)$. Lastly assume that $n \equiv r \not\equiv 0 \pmod{m}$. Then $r \in C$ or $m < c_t \equiv r \pmod{m}$ for exactly one index $t \in \{1, k\}$. Thus $\tau_1(C)$ consists of multiples of m while $\tau_2(C)$ still contains one part $\not\equiv 0 \pmod{m}$. In either case $C \in SP(n,m)$. Hence $H(n,m) \subseteq SP(n,m)$. The two sets are identical. □

As an illustration of Theorem 2 note $(2,9,4),(1,4,2,8) \notin SP(15,2)$ because the sequence of max m-powers of the parts are not unimodal, and $(2,10,3),(3,4,6,2) \notin SP(15,2)$ because the max m-powers are not distinct.

Theorem 3. *Let n,m be integers with $n \geq 0$, $m > 1$. Then*

$$sp(nm+1,m) = sp(nm+2,m) = \cdots = sp(nm+m-1,m) = 1 + 2\sum_{j=1}^{n} sp(j,m).$$

Proof. We first establish all but the last equality. By definition, $sp(1,m) = sp(2,m) = \cdots = sp(m-1,m) = 1$, and since $sp(m,m) = 1$, we have $sp(m+1,m) = 2sp(m,m) + sp(1,m) = 3$. Similarly $sp(m+2,m) = 3 = sp(m+3,m) = \cdots = sp(2m-1,m)$. Assume that the result holds for all integers $< nm$. Then with $1 \leq r \leq m-1$ we have $sp(nm+r,m) = 2sp(nm,m) + sp(nm-(m-r),m)$. But $1 \leq r \leq m-1 \implies m-1 \geq m-r \geq 1$ and $sp(nm-(m-r),m)$ is constant by the inductive hypothesis. Hence the result follows.

For the last equality, we iterate the recurrence (1.1). For each $1 \leq r \leq m-1$,

$$\begin{aligned} sp(mv+r,m) &= 2sp(mv,m) + sp(m(v-1)+r,m) \\ &= 2sp(v,m) + 2sp(v-1,m) + sp(m(v-2)+r,m) \\ &= \cdots \\ &= 2sp(v,m) + 2sp(v-1,m) + \cdots + 2sp(2,m) + sp(m+r,m). \end{aligned}$$

Since $sp(m+r,m) = 2sp(m,m) + sp(r,m) = 2sp(1,m) + sp(r,m)$, we obtain the desired result

$sp(mv+r,m) = 2sp(v,m) + 2sp(v-1,m) + \cdots + 2sp(2,m) + 2sp(1,m) + 1.$ □

Corollary 1. Given integers $m \geq 2$, then for any $j \geq 0$ and a fixed $v \in \{0,1,\ldots,m\}$,

$$sp(m^j(mv+r),m) = 2v+1,\ 1 \leq r \leq m-1.$$

Proof. By applying j-times of the first part of the recurrence (1.1), we obtain $sp(m^j(mv+r),m) = sp(mv+r,m)$. The last equality in Theorem 3 gives

$$sp(mv+r,m) = 1 + 2\sum_{i=1}^{v} sp(i,m),\ 0 \leq v \leq m,\ 1 \leq r < m.$$

Using (1.1), for $1 \leq i \leq m-1$, $sp(i,m) = 1$ and $sp(m,m) = 1$. Then we obtain $sp(mv+r,m) = 1 + 2\sum_{i=1}^{v} 1 = 1 + 2v$. Lastly, when $v = 0$, we have $sp(mv+r,m) = sp(r,m) = 1 + 2(0) = 1$, as expected. □

We note the following interesting case $v = 0$ of Corollary 1.

Corollary 2. Given an integer $m \geq 2$. For $1 \leq h \leq m-1$ and $i \geq 0$, we have

$$sp(m^i h, m) = 1 \quad \text{and} \quad sp(m^i, m) = 1.$$

4. Arithmetic Properties

The following two lemmas characterize the values of $oc(n,m)$ to a large extent.

Lemma 4. $sp(n,m)$ is odd for all integers $n \geq 0$, $m > 1$.

Proof. Use the bijection with $OC(n,m)$. Any n has exactly one OC composition into exactly one size of power of m, and all other OC compositions may be paired by whether they have their smallest part on the left or right. □

Vice versa, the following result is easily deduced from the first part of the relation (1.1).

Corollary 3. Given any nonnegative integer v, and $1 \leq r < m$, we have

$$sp(m^j(mv+r), m) = sp(mv+r, m) \; \forall \; j \geq 0.$$

We found the following pair of infinite modulo 4 congruences.

Theorem 4. *Given any integer $m \geq 2$. For all $j \geq 0$, we have*
(i) $sp(2mj+1, m) \equiv 1 \pmod 4$,
(ii) $sp(2mj+m+1, m) \equiv 3 \pmod 4$.

Proof. By induction on j. Note that $sp(1,m) = 1$ and $sp(m+1,m) = 2sp(m,m) + sp(1) = 3$. So the assertion holds for $j = 0$. Assume that (i) and (ii) hold for all $r < j$.

Then we first obtain

$$\begin{aligned} sp(2mj+1, m) &= 2sp(2mj, m) + sp(2mj+1-m, m) \\ &= 2sp(2j, m) + sp(2m(j-1)+m+1, m). \end{aligned}$$

Then by Lemma 4 $sp(2j,m)$ is odd, say $2u+1$, and the inductive hypothesis gives $sp(2m(j-1)+m+1, m) \equiv 3 \pmod 4$, say $4t+3$. Hence

$$sp(2mj+1, m) = 2(2u+1) + (4t+3) = 4(u+t) + 5 \equiv 1 \pmod 4$$

which proves part (i).

To prove part (ii) we have

$$\begin{aligned} sp(2mj+m+1, m) &= 2sp(2mj+m, m) + sp(2mj+1, m) \\ &= 2sp(2j+1, m) + sp(2mj+1, m). \end{aligned}$$

Then by Lemma 4 $sp(2j+1,m)$ is odd, say $2u+1$, and part (i) gives $sp(2mj+1,m) \equiv 1 \pmod 4$, say $4t+1$. Hence

$$sp(2mj+m+1, m) = 2(2u+1) + (4t+1) = 4(u+t) + 3 \equiv 3 \pmod 4.$$

□

Congruences modulo powers of 2 might be expected given the structure of the sets under discussion, with 2 options at a time for many choices. Perhaps more surprising, then, are the following congruences modulo 3:

(1) $sp(16j+5,4) \equiv 0 \pmod 3$;

(2) $sp(49j+8,7) \equiv 0 \pmod 3$;

(3) $sp(100j+11,10) \equiv 0 \pmod 3$;

. . . .

These three congruences are contained in the following infinite modulo 3 congruence.

Theorem 5. *Given an integer $m \geq 4$ such that $m \equiv 1 \pmod 3$. For all $1 \leq r < m$, we have*

$$sp(m^2 j + m + r, m) \equiv 0 \pmod 3 \ \forall\, j \geq 0.$$

We give two proofs below.

First Proof of Theorem 5. We attack the problem from the $oc(n,m)$ characterization as one-place m-power compositions with multiplicities not divisible by m. For convenience we will denote these OC_m compositions.

Begin by noting one group of OC_m compositions: valid compositions include $(1^{m^2 j+m+r})$, $(1^r, m^{mj+1})$, and $(m^{mj+1}, 1^r)$, for three.

Next, consider all those compositions that include only m (and no higher powers of m), and 1. The number of m in the composition determines the number of 1s.

Note that mj is not a valid number of m in the composition, since it is divisible by m; however, $mj-1, mj-2, \ldots, mj-(m-1)$ are all valid numbers of m, and there is always a number of 1 congruent to r mod m with such choices. Hence there are $2(m-1)$ such compositions, and since $m \equiv 1 \pmod 3$, this collection numbers a multiple of 3.

Since $mj-m$ is not a valid number of m in the composition, but $mj-m-1, \ldots, mj-(2m-1)$ are, similar collections occur until we are down to one part of size m.

Thus compositions in which only parts of size 1 and m occur contribute a multiple of 3 to the total number of compositions.

Now consider any valid choice of numbers and orderings of powers m^2, m^3, etc. Suppose that these form a composition of Cm^2. The remaining value to be composed is $m^2(j-C)+m+r$. In particular, a number of 1s congruent to r mod m must be in the composition, and some number of m^1 ranging from 1 up to $m(j-C)+1$ will be in the composition.

For any valid choice of numbers and arrangement of the powers m^i with $i \geq 2$, we now make a similar argument to the "empty" case before. Group the six partitions in which there are no m^1 and the required 1s are on either side of composition, or the four possible arrangements in which there are $m(j-C)+1$ of m^1 and exactly r of 1. Of the other permissible numbers of m^1 and 1, there are four valid compositions for each, and there are a multiple of 3 such groups of compositions, as previously argued. Thus the total number of OC_m compositions of $m^2 j + m + r$ is 0 mod 3, as claimed. □

This argument can likely be generalized to additional congruences.

In order to give a second proof of the theorem, we first prove a crucial lemma.

Lemma 5. If $m \equiv 1 \pmod 3$, then for any integer $j \geq 0$,

$$\sum_{i=1}^{mj+1} sp(i,m) \equiv 1 \pmod 3.$$

Proof.

$$\begin{aligned}\sum_{i=1}^{mj+1} sp(i,m) &= \sum_{r=1}^{m-1} sp(r,m) + sp(m,m) + \sum_{r=1}^{m-1} sp(m+r,m) + sp(2m,m)\\ &\quad + \sum_{r=1}^{m-1} sp(2m+r,m) + \cdots + \sum_{r=1}^{m-1} sp(m(j-1)+r,m)\\ &\quad + sp(mj,m) + sp(mj+1,m)\\ &= \sum_{t=1}^{j} sp(mt,m) + sp(mj+1,m) + \sum_{t=0}^{j-1}\sum_{r=1}^{m-1} sp(mt+r,m).\end{aligned}$$

Then using Equation (1.1) and Theorem 3 we obtain

$$\begin{aligned}\sum_{i=1}^{mj+1} sp(i,m) &= \frac{sp(mj+1,m)-1}{2} + sp(mj+1,m) + \sum_{t=0}^{j-1}\sum_{r=1}^{m-1} sp(mt+r,m)\\ &= \frac{1}{2}\left(3sp(mj+1,m)-1\right) + \sum_{t=0}^{j-1}\sum_{r=1}^{m-1} sp(mt+r,m). \qquad (4.1)\end{aligned}$$

But

$$\begin{aligned}E_j(m) := \sum_{t=0}^{j-1}\sum_{r=1}^{m-1} sp(mt+r,m) &= m-1+\sum_{t=1}^{j-1}\sum_{r=1}^{m-1} sp(mt+r,m)\\ &= m-1+\sum_{t=1}^{j-1}\left(2\sum_{r=1}^{m-1} sp(mt,m) + \sum_{r=1}^{m-1} sp(mt+r-m,m)\right) \text{ (by Eq. (1.1))}\\ &= m-1+2(m-1)\sum_{t=1}^{j-1} sp(t,m) + \sum_{t=0}^{j-2}\sum_{r=1}^{m-1} sp(mt+r,m)\\ &= (m-1)sp(m(j-1),m) + \sum_{t=0}^{j-2}\sum_{r=1}^{m-1} sp(mt+r,m) \text{ (by Th. 3)}.\end{aligned}$$

Therefore

$$E_j(m) = (m-1)sp(m(j-1),m) + E_{j-1}(m). \qquad (4.2)$$

Iterating Equation (4.2) we obtain

$$E_j(m) = (m-1)\sum_{h=1}^{u} sp(m(j-h),m) + E_{j-u}(m),\ 1 \le u \le j-1.$$

In particular, the case $u = j-1$ with $E_1(m) = m-1 \equiv 0 \pmod 3$, implies

$$E_j(m) \equiv 0 \pmod 3.$$

Consequently, reducing Equation (4.1) modulo 3 gives

$$\sum_{i=1}^{mj+1} sp(i,m) \equiv \frac{1}{2}(0-1) + 0 \equiv 1 \pmod 3$$

which is the desired result. □

Second Proof of Theorem 5. By Theorem 3,

$$sp(m^2j+m+r,m) = sp(m(mj+1)+r,m) = 1 + 2\sum_{i=1}^{mj+1} sp(i,m).$$

From Lemma 5 the sum is congruent to 1 modulo 3, say $3v+1$ for some v. Hence

$$sp(m^2j+m+r,m) = 1 + 2(3v+1) \equiv 0 \pmod 3.$$

This completes the proof. □

5. The $m = 2$ case: Semi-Pell Compositions and Binary Compositions

When $m = 2$, the numbers $sp(n,2)$ coincide with the semi-Pell sequence $sp(n)$, since they have the same recurrence and initial conditions (see Equations (1.1) and (1.2)). The generating function for semi-Pell compositions is then (from Equation (2.2))

$$Q_2(x) = \sum_{i=0}^{\infty} \frac{x^{2^i}}{1-x^{2^{i+1}}} \prod_{t=0}^{i-1}\left(1 + \frac{2x^{2^t}}{1-x^{2^{t+1}}}\right).$$

It may be observed that in this case, our structural results *completely* characterize the sequence mod 4.

Theorem 4 implies that residues mod 4 actually match for the $m = 2$ case when the argument is odd: $sp(4m+i,2) \equiv i \pmod 4$, i.e., the values of $sp(n,2)$ and n agree at odd values of n modulo 4. This, combined with Corollary 3, gives a complete characterization of the residues of $sp(n,2)$:

Theorem 6. *Writing n uniquely as $n = 2^j(4k+i)$ with $i \in \{1,3\}$, it follows that $sp(n,2) \equiv i \pmod 4$.*

Andrews in [2] denotes by $ob(n)$ the number of partitions of n into powers of 2, in which each part size appears an odd number of times. Theorem 6 has the corollary that, for $n \equiv 2i+1 \pmod 4$, the number of these in which exactly 2 part sizes appear is congruent to i mod 2, for each of these correspond to exactly two compositions enumerated by $oc(n)$ by reordering, while every n has 1 additional such partition (and composition) into exactly 1 part size, and those into three or more part sizes correspond to a multiple of four such compositions.

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