ELEMENTARY PROOF OF CONGRUENCES INVOLVING TRINOMIAL COEFFICIENTS FOR BABBAGE AND MORLEY

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Abstract. The aim of this work is to establish congruences $\left(\bmod p^2\right)$ involving the trinomial coefficients $\binom{np-1}{p-1}_2$ and $\binom{np-1}{(p-1)/2}_2$ arising from the expansion of the powers of the polynomial $1+x+x^2$. In main results we extend some known congruences involving the binomial coefficients $\binom{np-1}{p-1}$ and $\binom{np-1}{(p-1)/2}$ and establish congruences link binomial coefficients and harmonic numbers.

1. Introduction and main results

Many mathematicians studied in the 19-th century congruences of the forms $\binom{2p-1}{p-1}$ and $\binom{p-1}{(p-1)/2}$. In 1819, Babbage [1] showed, for any prime number $p \geq 3$, the congruence

$$\binom{2p-1}{p-1} \equiv 1 \left(\bmod p^2\right).$$

In 1862, Wolstenholme [18] proved, for any prime number $p \geq 5$, that the above congruence can be extended to

$$\binom{2p-1}{p-1} \equiv 1 \left(\bmod p^3\right).$$

In 1895, Morley [15] proved, for any prime number $p \geq 5$, that

$$\begin{aligned}\binom{p-1}{(p-1)/2} &\equiv (-1)^{(p-1)/2} 4^{p-1} \\ &= (-1)^{(p-1)/2} (1+pq_2)^2 \left(\bmod p^3\right),\end{aligned}$$

where q_a is the Fermat quotient defined for a given prime number p by

$$q_a = q_a(p) := \frac{a^{p-1}-1}{p}, \quad a \in \mathbb{Z}-p\mathbb{Z},$$

and $\mathbb{Z}$ denotes the set of the integer numbers.
In 1900, Glaisher [9] proved, for any prime number $p \geq 5$, that the above congruence can also be extended to

$$\binom{np-1}{p-1} \equiv 1 \left(\bmod p^3\right), \ n \geq 1.$$

Date: August 12, 2024.

2020 *Mathematics Subject Classification.* Primary: 11A07; Secondary: 11B65, 05A10.

Key words and phrases. Binomial coefficients, trinomial coefficients, harmonic numbers, congruences.

Also, in 1953, Carlitz [6, 7] improved, for any prime number $p \geq 5$, Morley's congruence to

$$(-1)^{\frac{p-1}{2}} \binom{p-1}{(p-1)/2} \equiv 4^{p-1} + \frac{p^3}{12} \left(\bmod p^4\right).$$

Many mathematicians have been interested to generalize the congruence of Wostenhlom and Morly, such the works of Zhao [19], McIntosh [13], Meštrović [14], Bencherif et al. [3] and Sun [16]. Recently, Sun [17] gave some properties and congruences involving the coefficients $\binom{n}{k}_2$ defined by

$$\left(1+x+x^2\right)^n = \sum_{k=0}^{2n} \binom{n}{k}_2 x^k.$$

See also Cao & Pan [4] and Cao & Sun [5].
The ring of p-integers $\mathbb{Z}_{(p)}$ is the set of rational numbers whose denominator is not divisible by p. For all integers x and y of $\mathbb{Z}_{(p)}$ and for any prime number p, we say that x is congruent to y modulo p and to write then

$$x \equiv y \ (\bmod p).$$

if and only if we have

$$\operatorname{num}(x-y) \in p\mathbb{Z}.$$

The idea of this work is inspired from the congruences given by Wolstenholme and Morly. We study congruences modulo p^2 for the trinomial coefficients $\binom{np-1}{p-1}_2$ and $\binom{np-1}{(p-1)/2}_2$. We prove congruences involving trinomial coefficients, binomial coefficients and harmonic numbers.
Our main results are given as follows.

Theorem 1.1. *Let $p \geq 5$ be a prime number and n be a positive integer. We have*

$$\binom{np-1}{p-1}_2 \equiv \begin{cases} 1 + npq_3 \ \left(\bmod p^2\right) & \text{if } p \equiv 1 \ (\bmod 3), \\ -1 - npq_3 \ \left(\bmod p^2\right) & \text{if } p \equiv 2 \ (\bmod 3). \end{cases} \tag{1}$$

and

$$\binom{np-1}{\frac{p-1}{2}}_2 \equiv \begin{cases} 1 + np\left(2q_2 + \frac{1}{2}q_3\right) \ \left(\bmod p^2\right) & \text{if } p \equiv 1 \ (\bmod 6), \\ -\frac{1}{2}npq_3 \ \left(\bmod p^2\right) & \text{if } p \equiv 5 \ (\bmod 6). \end{cases} \tag{2}$$

Theorem 1.2. *For every prime number $p \geq 5$ we have*

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k} H_k \equiv \begin{cases} -q_3 \ (\bmod p) & \text{if } p \equiv 1 \ (\bmod 3), \\ q_3 \ (\bmod p) & \text{if } p \equiv 2 \ (\bmod 3), \end{cases} \tag{3}$$

and

$$\sum_{k=1}^{\left[\frac{p-1}{4}\right]} \binom{4k}{2k} \left(\frac{2H_{2k}-H_k}{4^k}\right) \equiv \begin{cases} -(-1)^{\frac{p-1}{2}} \frac{q_3}{2} \ (\bmod p) & \text{if } p \equiv 1 \ (\bmod 6), \\ (-1)^{\frac{p-1}{2}} \frac{q_3}{2} \ (\bmod p) & \text{if } p \equiv 5 \ (\bmod 6), \end{cases} \tag{4}$$

where H_n to be the n-th harmonic number defined by

$$H_0 = 0, \quad H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Proposition 1.3. *Let $p \geq 5$ be a prime number and n be a positive integer. Then*

$$\sum_{k=0}^{p-1} \binom{np-1}{k}_2 \equiv \begin{cases} 1 + npq_3 \pmod{p^2} & \text{if } p \equiv 1 \pmod 3, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod 3 \end{cases} \tag{5}$$

and

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{np-1}{k}_2 \equiv \begin{cases} 1 + np\left(\frac{4}{3}q_2 + q_3\right) \pmod{p^2} & \text{if } p \equiv 1 \pmod 6, \\ -\frac{2}{3}npq_2 \pmod{p^2} & \text{if } p \equiv 5 \pmod 6. \end{cases} \tag{6}$$

For $k \leq p-1$, since

$$\binom{np-1}{k} = (-1)^k \prod_{i=1}^{k} \left(1 - \frac{np}{i}\right) \equiv (-1)^k (1 - npH_k) \pmod{p^2} \tag{7}$$

we conclude that $\binom{np^2-1}{k} \equiv (-1)^k \pmod{p^2}$.
A similar congruence for the coefficients $\binom{np^2-1}{k}_2$ is given as follows:

Corollary 1.4. *Let $p \geq 5$ be a prime number and n, k be integers with $n \geq 1$ and $k \in \{0, 1, \ldots, p-1\}$. We have*

$$\binom{np^2-1}{k}_2 \equiv \begin{cases} 1 \pmod{p^2} & \text{if } k \equiv 0 \pmod 3, \\ -1 \pmod{p^2} & \text{if } k \equiv 1 \pmod 3, \\ 0 \pmod{p^2} & \text{if } k \equiv 2 \pmod 3. \end{cases} \tag{8}$$

2. Some basic congruences

In this section, we give some congruences involving harmonic numbers and trinomial coefficients in order to prove the main theorems.

Lemma 2.1. [8, 10, 12] *Let p be a prime number. We have*

$$H_{[p/2]} \equiv -2q_2 \pmod p, \quad p \geq 3, \tag{9}$$

$$H_{[p/3]} \equiv -\frac{3}{2}q_3 \pmod p, \quad p \geq 5, \tag{10}$$

$$H_{[p/6]} \equiv -2q_2 - \frac{3}{2}q_3 \pmod p, \quad p \geq 5. \tag{11}$$

Lemma 2.2. *For any prime number $p \geq 3$ we have*

$$H_{p-k} \equiv H_{k-1} \pmod p, \quad 1 \leq k \leq p-1, \tag{12}$$

$$H_{\frac{p-1}{2}-k} \equiv -2q_2 + 2H_{2k} - H_k \pmod p, \; 1 \leq k \leq \frac{p-1}{2}. \tag{13}$$

Proof. When $k \in \{1, 2, \ldots, p-1\}$, we have

$$\begin{aligned} H_{p-k} &= \sum_{i=1}^{p-k} \frac{1}{i} \\ &= \sum_{i=1}^{p-1} \frac{1}{i} - \sum_{i=p-k+1}^{p-1} \frac{1}{i} \\ &= H_{p-1} - \sum_{i=1}^{k-1} \frac{1}{p-k+i}. \end{aligned}$$

Then, since $H_{p-1} \equiv 0 \pmod p$ we get

$$H_{p-k} \equiv \sum_{i=1}^{k-1} \frac{1}{k-i} = H_{k-1} \pmod p.$$

Similarly, if $k \in \{1, 2, \ldots, (p-1)/2\}$, we get

$$\begin{aligned} H_{(p-1)/2-k} &= \sum_{j=1}^{(p-1)/2-k} \frac{1}{j} \\ &= \sum_{j=1}^{(p-1)/2} \frac{1}{j} - \sum_{j=(p-1)/2-k+1}^{(p-1)/2} \frac{1}{j} \\ &= H_{(p-1)/2} - \frac{2}{p-1} - \sum_{j=(p-1)/2-k+1}^{(p-3)/2} \frac{1}{j} \\ &= H_{(p-1)/2} - \frac{2}{p-1} - \sum_{j=1}^{k-1} \frac{1}{(p-1)/2-j} \\ &= H_{(p-1)/2} - \frac{2}{p-1} - \sum_{j=1}^{k-1} \frac{2}{p-1-2j}, \end{aligned}$$

and since $H_{(p-1)/2} \equiv -2q_2 \pmod p$ [8], we conclude that

$$\begin{aligned} H_{(p-1)/2-k} &\equiv -2q_2 + 2 + 2\sum_{j=1}^{k-1} \frac{1}{2j+1} \\ &= -2q_2 + 2 + 2\left(\sum_{j=1}^{2k-1} \frac{1}{j} - \sum_{j=1}^{k-1} \frac{1}{2j} - 1\right) \\ &= -2q_2 + 2H_{2k-1} - H_{k-1} \\ &= -2q_2 + 2H_{2k} - H_k \pmod p. \end{aligned}$$

∎

Lemma 2.3. *Let $p \geq 5$ be a prime number. Then, if $p \equiv 1 \pmod 3$ we obtain*

$$\sum_{k=0}^{(p-4)/3} \frac{1}{3k+2} \equiv 0 \pmod{p}, \tag{14}$$

$$\sum_{k=0}^{(p-4)/3} \frac{1}{3k+1} \equiv \frac{1}{2} q_3 \pmod{p}, \tag{15}$$

and if $p \equiv 2 \pmod 3$ we obtain

$$\sum_{k=0}^{(p-5)/3} \frac{1}{3k+1} \equiv 1 \pmod{p}, \tag{16}$$

$$\sum_{k=0}^{(p-5)/3} \frac{1}{3k+2} \equiv \frac{1}{2} q_3 \pmod{p}. \tag{17}$$

Proof. For any prime number $p \equiv 1 \pmod 3$, we have

$$\begin{aligned} \sum_{k=0}^{(p-4)/3} \frac{1}{3k+2} &= \sum_{k=0}^{(p-4)/3} \frac{1}{3\left(\frac{p-4}{3} - k\right) + 2} \\ &\equiv -\sum_{k=0}^{(p-4)/3} \frac{1}{2+3k} \pmod{p} \end{aligned}$$

and this gives the congruence (14). From the identity

$$\sum_{k=0}^{(p-4)/3} \frac{1}{3k+1} + \sum_{k=0}^{(p-4)/3} \frac{1}{3k+2} + \sum_{k=0}^{(p-4)/3} \frac{1}{3k+3} = \sum_{k=1}^{p-1} \frac{1}{k}$$

and by the congruences $H_{p-1} \equiv 0 \pmod{p}$ and (10) it results

$$\begin{aligned} \sum_{k=0}^{(p-4)/3} \frac{1}{3k+1} &= H_{p-1} - \frac{1}{3} H_{[p/3]} - \sum_{k=0}^{(p-4)/3} \frac{1}{3k+2} \\ &\equiv 0 + \frac{1}{2} q_3 - 0 \\ &\equiv \frac{1}{2} q_3 \pmod{p} \end{aligned}$$

which gives the congruence (15).
Also, if $p \equiv 2 \pmod 3$, the other congruences can be proved similarly. ∎

Lemma 2.4. *For any prime number $p \geq 5$ we have*

$$\sum_{k=0}^{(p-1)/6} \frac{1}{2k+1} \equiv q_2 - \frac{3}{4} q_3 + \frac{3}{2} \pmod{p} \quad \text{if } p \equiv 1 \pmod 6, \tag{18}$$

$$\sum_{k=0}^{(p-5)/6} \frac{1}{2k+1} \equiv q_2 - \frac{3}{4} q_3 \pmod{p} \quad \text{if } p \equiv 5 \pmod 6. \tag{19}$$

Proof. For $p \equiv 1 \pmod 6$ use the congruence (10) to obtain

$$\begin{aligned}\sum_{k=0}^{(p-1)/6} \frac{1}{2k+1} + \sum_{k=1}^{(p-1)/6} \frac{1}{2k} &= \sum_{k=1}^{(p-1)/3+1} \frac{1}{k} = H_{[p/3]} + \frac{3}{p+2} \\ &\equiv -\frac{3}{2}q_3 + \frac{3}{2} \pmod p,\end{aligned}$$

and for $p \equiv 5 \pmod 6$ use the congruence (10) to obtain

$$\sum_{k=0}^{(p-5)/6} \frac{1}{2k+1} + \sum_{k=1}^{(p-5)/6} \frac{1}{2k} = \sum_{k=1}^{(p-2)/3} \frac{1}{k} = H_{[p/3]} \equiv -\frac{3}{2}q_3 \pmod p.$$

So, by (11) it results

$$\sum_{k=0}^{[p/6]} \frac{1}{2k} = \frac{1}{2}H_{[p/6]} \equiv -q_2 - \frac{3}{4}q_3 \pmod p,$$

from which the desired congruences follow. ∎

Lemma 2.5. *Let p be a prime number.*
Then, for $p \equiv 1 \pmod 6$ we have

$$\sum_{k=0}^{(p-1)/6} \frac{1}{3k+1} \equiv -\frac{2}{3}q_2 + 2 \pmod p, \tag{20}$$

$$\sum_{k=0}^{(p-1)/6} \frac{1}{3k+2} \equiv -\frac{2}{3}q_2 + \frac{1}{2}q_3 + \frac{2}{3} \pmod p, \tag{21}$$

and, for $p \equiv 5 \pmod 6$ we have

$$\sum_{k=0}^{(p-5)/6} \frac{1}{3k+1} \equiv \frac{1}{2}q_3 - \frac{2}{3}q_2 \pmod p, \tag{22}$$

$$\sum_{k=0}^{(p-5)/6} \frac{1}{3k+2} \equiv -\frac{2}{3}q_2 \pmod p. \tag{23}$$

Proof. For $p \equiv 1 \pmod{6}$, by the congruences (17) and (18) we get

$$\begin{aligned}
\sum_{k=0}^{(p-1)/6} \frac{1}{3k+2} &= 2\sum_{k=0}^{(p-1)/6} \frac{1}{6k+4} \\
&= 2\sum_{k=0}^{(p-1)/3} \frac{1}{6k+4} - 2\sum_{k=(p-1)/6+1}^{(p-1)/3} \frac{1}{6k+4} \\
&= \sum_{k=0}^{(p-1)/3} \frac{1}{3k+2} - 2\sum_{k=1}^{(p-1)/6} \frac{1}{6k+p+3} \\
&\equiv 1 - \frac{2}{3}\sum_{k=1}^{(p-1)/6} \frac{1}{2k+1} \\
&\equiv 1 - \frac{2}{3}\left(q_2 - \frac{3}{4}q_3 + \frac{1}{2}\right) \\
&= -\frac{2}{3}q_2 + \frac{1}{2}q_3 + \frac{2}{3} \pmod{p}.
\end{aligned}$$

We also have

$$\sum_{k=1}^{(p+5)/2} \frac{1}{k} = \sum_{k=0}^{(p-1)/6} \frac{1}{3k+1} + \sum_{k=0}^{(p-1)/6} \frac{1}{3k+2} + \frac{1}{3}\sum_{k=0}^{(p-1)/6} \frac{1}{k+1}$$

which gives on using the congruences (9), (11) and (21)

$$\begin{aligned}
&\sum_{k=0}^{(p-1)/6} \frac{1}{3k+1} \\
&\equiv \sum_{k=1}^{(p+5)/2} \frac{1}{k} - \sum_{k=0}^{(p-1)/6} \frac{1}{3k+2} - \frac{1}{3}\sum_{k=0}^{(p-1)/6} \frac{1}{k+1} \\
&= H_{[p/2]} + \frac{2}{p+5} + \frac{2}{p+3} + \frac{2}{p+1} \\
&- \sum_{k=0}^{(p-1)/6} \frac{1}{3k+2} - \frac{1}{3}\left(H_{[p/6]} + \frac{6}{p+5}\right) \\
&\equiv -2q_2 + \frac{8}{3} - \left(-\frac{2}{3}q_2 + \frac{1}{2}q_3 + \frac{2}{3}\right) - \frac{1}{3}\left(-2q_2 - \frac{3}{2}q_3\right) \\
&= -\frac{2}{3}q_2 + 2 \pmod{p}.
\end{aligned}$$

For $p\equiv 5\ (\text{mod}6)$ use the congruence (17) to get

$$\begin{aligned}\sum_{k=0}^{(p-5)/6}\frac{1}{3k+2} &= 2\sum_{k=0}^{(p-5)/3}\frac{1}{6k+4}-2\sum_{k=(p-5)/6+1}^{(p-5)/3}\frac{1}{6k+4}\\ &= \sum_{k=0}^{(p-5)/3}\frac{1}{3k+2}-2\sum_{k=1}^{(p-5)/6}\frac{1}{6k+p-1}\\ &\equiv \frac{1}{2}q_3-2\sum_{k=1}^{(p-5)/6}\frac{1}{6k-1}\ (\text{mod}p).\end{aligned}$$

by setting $k=(p+1)/6-j$ and using (11) this last congruence becomes

$$\sum_{k=1}^{(p-5)/6}\frac{1}{6k-1}\equiv-\frac{1}{6}\sum_{j=1}^{(p-5)/6}\frac{1}{j}=-\frac{1}{6}H_{[p/6]}\equiv\frac{1}{3}q_2+\frac{1}{4}q_3\ (\text{mod}p),$$

hence $\sum_{k=0}^{(p-5)/6}\frac{1}{3k+2}\equiv\frac{1}{2}q_3-2\left(\frac{1}{3}q_2+\frac{1}{4}q_3\right)\equiv-\frac{2}{3}q_2\ (\text{mod}p)$. We also have

$$\sum_{k=0}^{(p-5)/6}\frac{1}{3k+1}+\sum_{k=0}^{(p-5)/6}\frac{1}{3k+2}+\frac{1}{3}\sum_{k=0}^{(p-5)/6}\frac{1}{k+1}=\sum_{k=1}^{(p+1)/2}\frac{1}{k}$$

and by using the congruences (9), (10) and (23) this gives

$$\begin{aligned}&\sum_{k=0}^{(p-5)/6}\frac{1}{3k+1}\\ &= \left(\frac{2}{p+1}+H_{[p/2]}\right)-\frac{1}{3}\left(\frac{6}{p+1}+H_{[p/6]}\right)-\sum_{k=0}^{(p-5)/6}\frac{1}{3k+2}\\ &\equiv \frac{1}{2}q_3-\frac{2}{3}q_2\ (\text{mod}p).\end{aligned}$$

∎

Proposition 2.6. *Let $p\geq 5$ be a prime number and n,k be positive integers. We have*

$$\binom{np-1}{3k}_2\equiv 1-np\left(\frac{2}{3}H_k+\sum_{j=0}^{k-1}\frac{1}{3j+2}\right)\ \left(\text{mod}p^2\right),\ k\leq\tfrac{p-1}{3},\tag{24}$$

$$\binom{np-1}{3k+1}_2\equiv -1+np\left(\frac{2}{3}H_k+\sum_{j=0}^{k}\frac{1}{3j+1}\right)\ \left(\text{mod}p^2\right),\ k\leq\tfrac{p-2}{3},\tag{25}$$

$$\binom{np-1}{3k+2}_2\equiv np\left(-\sum_{j=0}^{k}\frac{1}{3j+1}+\sum_{j=0}^{k}\frac{1}{3j+2}\right)\ \left(\text{mod}p^2\right),\ k\leq\tfrac{p-3}{3}.\tag{26}$$

Proof. From the expansion

$$\begin{aligned}\left(1+x+x^2\right)^n &= \left(x+e^{i\frac{\pi}{3}}\right)^n\left(x+e^{-i\frac{\pi}{3}}\right)^n\\ &= \sum_{k\geq 0}\left(\sum_{j=0}^{k}\binom{n}{j}\binom{n}{k-j}e^{(k-2j)i\frac{\pi}{3}}\right)x^k\end{aligned}$$

we deduce the identity

$$\binom{n}{k}_2 = \sum_{j=0}^{k}\binom{n}{j}\binom{n}{k-j}\cos\frac{(k-2j)\,\pi}{3}. \tag{27}$$

and by the congruence (7) and the identity (27) we get

$$\begin{aligned}&\binom{np-1}{k}_2\\ &= \sum_{j=0}^{k}\binom{np-1}{j}\binom{np-1}{k-j}\cos\frac{(k-2j)\,\pi}{3}\\ &\equiv (-1)^k\sum_{j=0}^{k}\left(1-np\left(H_j+H_{k-j}\right)\right)\cos\frac{(k-2j)\,\pi}{3}\\ &= (-1)^k\sum_{j=0}^{k}\left(1-2npH_j\right)\cos\frac{(k-2j)\,\pi}{3}\left(\bmod p^2\right)\end{aligned}$$

Then, for the congruence (24), we have

$$\begin{aligned}
\binom{np-1}{3k}_2 &\equiv \sum_{j=0}^{3k}\cos\frac{2j\pi}{3} - 2np\sum_{j=0}^{3k}H_j\cos\frac{2j\pi}{3}\\
&= 1-2np\sum_{j=0}^{3k}H_j\cos\frac{2j\pi}{3}\\
&= 1-np\left(2\sum_{j=0}^{k}H_{3j}-\sum_{j=0}^{k-1}H_{3j+1}-\sum_{j=0}^{k-1}H_{3j+2}\right)\\
&= 1-np\left(\sum_{j=0}^{k-1}\left(2H_{3j}-H_{3j+1}-H_{3j+2}\right)+2H_{3k}\right)\\
&= 1-2npH_{3k}+np\left(2\sum_{j=0}^{k-1}\frac{1}{3j+1}+\sum_{j=0}^{k-1}\frac{1}{3j+2}\right)\\
&= 1-2np\left(\sum_{j=0}^{k-1}\frac{1}{3j+1}+\sum_{j=0}^{k-1}\frac{1}{3j+2}+\sum_{j=1}^{k}\frac{1}{3j}\right)\\
&+np\left(2\sum_{j=0}^{k-1}\frac{1}{3j+1}+\sum_{j=0}^{k-1}\frac{1}{3j+2}\right)\\
&= 1-np\left(\frac{2}{3}H_k+\sum_{j=0}^{k-1}\frac{1}{3j+2}\right) \pmod{p^2}.
\end{aligned}$$

For the congruence (25) we have

$$\begin{aligned}
&\binom{np-1}{3k+1}_2\\
&\equiv -\sum_{j=0}^{3k+1}\cos\frac{(2j-1)\pi}{3}+2np\sum_{j=0}^{3k+1}H_j\cos\frac{(2j-1)\pi}{3}\\
&= -1+2np\sum_{j=0}^{3k+1}H_j\cos\frac{(2j-1)\pi}{3}\\
&= -1+np\left(\sum_{j=0}^{k}\left(H_{3j}+H_{3j+1}-2H_{3j+2}\right)+2H_{3k+2}\right)
\end{aligned}$$

$$
\begin{aligned}
&= -1 + np\left(-\sum_{j=0}^{k}\frac{1}{3j+1} - 2\sum_{j=0}^{k}\frac{1}{3j+2}\right)\\
&+ np\left(2H_{3k} + \frac{2}{3k+1} + \frac{2}{3k+2}\right)\\
&= -1 + 2npH_{3k} + np\left(-\sum_{j=0}^{k-1}\frac{1}{3j+1} - 2\sum_{j=0}^{k-1}\frac{1}{3j+2} + \frac{1}{3k+1}\right)\\
&= -1 + 2np\left(\sum_{j=0}^{k-1}\frac{1}{3j+1} + \sum_{j=0}^{k-1}\frac{1}{3j+2} + \sum_{j=1}^{k}\frac{1}{3j}\right)\\
&+ np\left(-\sum_{j=0}^{k-1}\frac{1}{3j+1} - 2\sum_{j=0}^{k-1}\frac{1}{3j+2} + \frac{1}{3k+1}\right)\\
&= -1 + np\left(\frac{2}{3}H_k + \sum_{j=0}^{k}\frac{1}{3j+1}\right) \quad \left(\mathrm{mod} p^2\right).
\end{aligned}
$$

For the congruence (26) we have

$$
\begin{aligned}
\binom{np-1}{3k+2}_2 &\equiv \sum_{j=0}^{3k+2}\cos\frac{(2j-2)\pi}{3} - 2np\sum_{j=0}^{3k+2}H_j\cos\frac{(2j-2)\pi}{3}\\
&= -2np\sum_{j=0}^{3k+2}H_j\cos\frac{(2j-2)\pi}{3}\\
&= np\left(\sum_{j=0}^{k}\left(H_{3j} - 2H_{3j+1} + H_{3j+2}\right)\right)\\
&\equiv np\left(-\sum_{j=0}^{k}\frac{1}{3j+1} + \sum_{j=0}^{k}\frac{1}{3j+2}\right) \quad \left(\mathrm{mod} p^2\right).
\end{aligned}
$$

∎

3. Proof of the main results

Proof of Theorem 1.1. For $p \equiv 1 \pmod 3$ let $3k = p-1$ in the congruence (24). Then, by the congruences (10) and (14) we obtain

$$
\begin{aligned}
\binom{np-1}{p-1}_2 &\equiv 1 - np\left(\frac{2}{3}H_{(p-1)/3} + \sum_{k=0}^{(p-4)/3}\frac{1}{3j+2}\right)\\
&\equiv 1 + npq_3 \quad \left(\mathrm{mod} p^2\right).
\end{aligned}
$$

For $p\equiv 2 \pmod 3$ let $3k+1=p-1$ in the congruence (25). Then, by the congruences (10) and (16) we obtain

$$\begin{aligned}\binom{np-1}{p-1}_2 &\equiv -1+np\left(\frac{2}{3}H_{(p-2)/3}+\sum_{j=0}^{(p-2)/3}\frac{1}{3j+1}\right)\\ &\equiv -1-npq_3 \pmod{p^2}.\end{aligned}$$

For $p\equiv 1 \pmod 6$ let $3k=(p-1)/2$ in the congruence (24). Then, by the congruences (11) and (20) we obtain

$$\begin{aligned}\binom{np-1}{3k}_2 &\equiv 1-np\left(\frac{2}{3}H_k+\sum_{j=0}^{k-1}\frac{1}{3j+2}\right)\\ &\equiv 1+np\left(2q_2+\frac{1}{2}q_3\right) \pmod{p^2}.\end{aligned}$$

For $p\equiv 5 \pmod 6$ let $3k+2=(p-1)/2$ in the congruence (26). Then, by the congruences (22) and (23) we obtain

$$\begin{aligned}\binom{np-1}{(p-1)/2}_2 &\equiv np\left(-\sum_{j=0}^{(p-5)/6}\frac{1}{3j+1}+\sum_{j=0}^{(p-5)/6}\frac{1}{3j+2}\right)\\ &\equiv -\frac{1}{2}npq_3 \pmod{p^2}.\end{aligned}$$

∎

Proof of Theorem 1.2. By the known identity [2, Eq. 1.5]

$$\binom{np-1}{p-1}_2=\sum_{k=(p-1)/2}^{p-1}\binom{np-1}{k}\binom{k}{p-1-k} \tag{28}$$

we have

$$\begin{aligned}\binom{np-1}{p-1}_2 &= \sum_{k=(p-1)/2}^{p-1}\binom{np-1}{k}\binom{k}{p-1-k}\\ &\equiv \sum_{k=(p-1)/2}^{p-1}(-1)^k(1-npH_k)\binom{k}{p-1-k}\\ &= \sum_{k=0}^{(p-1)/2}(-1)^k\binom{p-1-k}{k}(1-npH_{p-1-k}) \pmod{p^2}.\end{aligned}$$

So, by the congruence

$$\begin{aligned}\binom{p-1-k}{k} &= \frac{\binom{p-1}{2k}}{\binom{p-1}{k}}\binom{2k}{k}\\ &\equiv (-1)^k\binom{2k}{k}\ (\bmod p)\,,\ k\in\left\{0,\ldots,\frac{p-1}{2}\right\},\end{aligned}$$

the identity [11, Cor. 2.8]

$$\sum_{k=0}^{[n/2]}(-1)^k\binom{n-k}{k}=\begin{cases}0 & \text{if } n\equiv 2\ (\bmod 3)\,,\\ (-1)^{[n/3]} & \text{otherwise}\end{cases} \tag{29}$$

and the congruence (12) we get

$$\binom{np-1}{p-1}_2\equiv(-1)^{[(p-1)/3]}-np\sum_{k=0}^{(p-1)/2}\binom{2k}{k}H_k\ \left(\bmod p^2\right).$$

We note here that $(-1)^{[(p-1)/3]}=1$ if $p\equiv 1\ (\bmod 3)$ and
$(-1)^{[(p-1)/3]}=-1$ if $p\equiv 2\ (\bmod 3)$.
Hence, by combining this congruence with the congruence (1), we obtain the congruence (3). By the identity [2, Eq. 1.5] we have

$$\begin{aligned}&\binom{np-1}{(p-1)/2}_2\\ &=\sum_{k=0}^{(p-1)/2}\binom{np-1}{k}\binom{k}{(p-1)/2-k}\\ &\equiv\sum_{k=0}^{(p-1)/2}(-1)^k(1-npH_k)\binom{k}{(p-1)/2-k}\\ &=(-1)^{(p-1)/2}\sum_{k=0}^{(p-1)/2}(-1)^k\binom{(p-1)/2-k}{k}\\ &-(-1)^{\frac{p-1}{2}}np\sum_{k=0}^{\frac{p-1}{2}}(-1)^k\binom{(p-1)/2-k}{k}H_{\frac{p-1}{2}-k}\ \left(\bmod p^2\right).\end{aligned}$$

Then, by the congruence (13) the last congruence can be written as

$$\begin{aligned}&\binom{np-1}{(p-1)/2}_2\\&\equiv(-1)^{\frac{p-1}{2}}(1+2npq_2)\sum_{k=0}^{\left[\frac{p-1}{4}\right]}(-1)^k\binom{\frac{p-1}{2}-k}{k}\\&-(-1)^{\frac{p-1}{2}}np\sum_{k=1}^{\left[\frac{p-1}{4}\right]}(-1)^k\binom{\frac{p-1}{2}-k}{k}(2H_{2k}-H_k)\pmod{p^2}.\end{aligned}$$

But for $k\in\{1,2,\ldots,[(p-1)/4]\}$ we have

$$\begin{aligned}&(-1)^k\binom{\frac{p-1}{2}-k}{k}\\&=(-1)^k\frac{(p-(2k+1))(p-(2k+3))\cdots(p-(4k-1))}{2^kk!}\\&\equiv\frac{(2k+1)(2k+3)\cdots(4k-1)}{2^kk!}\\&=\frac{1}{4^k}\binom{4k}{2k}\pmod{p},\end{aligned}$$

hence

$$\begin{aligned}&\binom{np-1}{(p-1)/2}_2\\&\equiv(-1)^{\frac{p-1}{2}}(1+2npq_2)\sum_{k=0}^{\left[\frac{p-1}{4}\right]}(-1)^k\binom{\frac{p-1}{2}-k}{k}\\&-(-1)^{\frac{p-1}{2}}np\sum_{k=1}^{\left[\frac{p-1}{4}\right]}\binom{4k}{2k}\left(\frac{2H_{2k}-H_k}{4^k}\right)\pmod{p^2}.\end{aligned}\tag{30}$$

Then, for $p\equiv1\pmod{6}$, the identity (29) shows that we have

$$\sum_{k=0}^{(p-1)/2}(-1)^k\binom{(p-1)/2-k}{k}=(-1)^{[(p-1)/6]},$$

and since $(-1)^{(p-1)/2+[(p-1)/6]}=1$, the congruence (30) becomes

$$\begin{aligned}&\binom{np-1}{\frac{p-1}{2}}_2\\&\equiv\ 1+npq_2-(-1)^{\frac{p-1}{2}}np\sum_{k=1}^{\left[\frac{p-1}{4}\right]}\frac{1}{4^k}\binom{4k}{2k}(2H_{2k}-H_k)\pmod{p^2},\end{aligned}$$

and by combining this congruence with the congruence (2), we obtain the congruence (4). Also, for $p \equiv 5 \pmod{6}$, the identity (29) shows that we have

$$\sum_{k=0}^{(p-1)/2} (-1)^k \binom{(p-1)/2-k}{k} = 0,$$

so the congruence (30) becomes

$$\begin{aligned} &\binom{np-1}{(p-1)/2}_2 \\ \equiv\ & -(-1)^{(p-1)/2} np \sum_{k=1}^{[(p-1)/4]} \frac{1}{4^k}\binom{4k}{2k}(2H_{2k}-H_k) \pmod{p^2}, \end{aligned}$$

and by combining this congruence with the congruence (2), we obtain the congruence (4). ∎

Proof of Proposition 1.3. For $k \in \{0, \ldots, [p/3]-1\}$, from Proposition 2.6 we may state

$$\binom{np-1}{3k}_2 + \binom{np-1}{3k+1}_2 + \binom{np-1}{3k+2}_2 \equiv \frac{np}{3k+2} \pmod{p^2},. \tag{31}$$

To prove the congruences (5) let

$$\begin{aligned} &\sum_{j=0}^{p-1}\binom{np-1}{j}_2 \\ =\ & \sum_{j=0}^{\left[\frac{p-1}{3}\right]}\binom{np-1}{3j}_2 + \sum_{j=0}^{\left[\frac{p-2}{3}\right]}\binom{np-1}{3j+1}_2 + \sum_{j=0}^{\left[\frac{p-3}{3}\right]}\binom{np-1}{3j+2}_2. \end{aligned}$$

For $p \equiv 1 \pmod{3}$, by the congruences (31), (1) and (14), we get

$$\begin{aligned} &\sum_{j=0}^{p-1}\binom{np-1}{j}_2 \\ &= \sum_{j=0}^{\frac{p-1}{3}}\binom{np-1}{3j}_2 + \sum_{j=0}^{\frac{p-1}{3}-1}\binom{np-1}{3j+1}_2 + \sum_{j=0}^{\frac{p-1}{3}-1}\binom{np-1}{3j+2}_2 \\ &\equiv \binom{np-1}{p-1}_2 + \sum_{j=0}^{\frac{p-4}{3}}\left[\binom{np-1}{3j}_2 + \binom{np-1}{3j+1}_2\binom{np-1}{3j+2}_2\right] \\ &\equiv \binom{np-1}{p-1}_2 + np\sum_{j=0}^{\frac{p-4}{3}}\frac{1}{3j+2} \\ &\equiv 1 + npq_3 \pmod{p^2}. \end{aligned}$$

For $p\equiv 2 \pmod 3$, by the congruences (31), (24), (10), (1) and (17), we get

$$\begin{aligned}
&\sum_{j=0}^{p-1}\binom{np-1}{j}_2\\
&=\sum_{j=0}^{\frac{p-2}{3}}\binom{np-1}{3j}_2+\sum_{j=0}^{\frac{p-2}{3}}\binom{np-1}{3j+1}_2+\sum_{j=0}^{\frac{p-2}{3}-1}\binom{np-1}{3j+2}_2\\
&\equiv\binom{np-1}{p-2}_2+\binom{np-1}{p-1}_2+np\sum_{j=0}^{\frac{p-5}{3}}\frac{1}{3j+2}\\
&\equiv 1-np\left(\frac{2}{3}H_{(p-2)/3}+\sum_{j=0}^{\frac{p-5}{3}}\frac{1}{3j+2}\right)+(-1-npq_3)+\left(\frac{npq_3}{2}\right)\\
&\equiv 1-np\left(\frac{2}{3}\left(-\frac{3}{2}q_3\right)+\frac{1}{2}q_3\right)-1-\frac{1}{2}npq_3\\
&=0\ \left(\mathrm{mod} p^2\right).
\end{aligned}$$

To prove the congruence (6) let

$$\begin{aligned}
&\sum_{j=0}^{(p-1)/2}\binom{np-1}{j}_2\\
=\ &\sum_{j=0}^{[(p-1)/6]}\binom{np-1}{3j}_2+\sum_{j=0}^{[(p-3)/6]}\binom{np-1}{3j+1}_2+\sum_{j=0}^{[(p-5)/6]}\binom{np-1}{3j+2}_2.
\end{aligned}$$

For $p\equiv 1 \pmod 6$, by the congruences (31), (1) and (21), we obtain

$$\begin{aligned}
&\sum_{k=0}^{(p-1)/2}\binom{np-1}{k}_2\\
&=\sum_{j=0}^{\frac{p-1}{6}}\binom{np-1}{3j}_2+\sum_{j=0}^{\frac{p-1}{6}-1}\binom{np-1}{3j+1}_2+\sum_{j=0}^{\frac{p-1}{6}-1}\binom{np-1}{3j+2}_2\\
&\equiv\tbinom{np-1}{\frac{p-1}{2}}_2+\sum_{j=0}^{\frac{p-7}{6}}\left[\tbinom{np-1}{3j}_2+\tbinom{np-1}{3j+1}_2+\tbinom{np-1}{3j+2}_2\right]\\
&\equiv\binom{np-1}{(p-1)/2}_2+np\sum_{j=0}^{(p-7)/6}\frac{1}{3j+2}\\
&\equiv 1+np\left(2q_2+\frac{1}{2}q_3\right)+np\left(-\frac{2}{3}q_2+\frac{1}{2}q_3\right)\\
&=1+np\left(\frac{4}{3}q_2+q_3\right)\ \ \left(\mathrm{mod} p^2\right).
\end{aligned}$$

For $p \equiv 5 \pmod 6$, by the congruences (31) and (21), we obtain

$$\begin{aligned}
&\sum_{k=0}^{(p-1)/2} \binom{np-1}{k}_2 \\
&= \sum_{j=0}^{(p-5)/6} \binom{np-1}{3j}_2 + \sum_{j=0}^{(p-5)/6} \binom{np-1}{3j+1}_2 + \sum_{j=0}^{(p-5)/6} \binom{np-1}{3j+2}_2 \\
&\equiv \sum_{j=0}^{(p-5)/6} \left[\binom{np-1}{3j}_2 + \binom{np-1}{3j+1}_2 + \binom{np-1}{3j+2}_2 \right] \\
&\equiv np \sum_{j=0}^{(p-5)/6} \frac{1}{3j+2} \equiv -\frac{2}{3} npq_2 \pmod{p^2}.
\end{aligned}$$

∎

Proof of Corollary 1.4. Corollary 1.4 follows easily from Proposition 2.6. ∎

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