ON TWO CONJECTURES DUE TO SUN

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ABSTRACT. We prove two conjectures due to Sun concerning binomial-harmonic sums. First, we introduce a proof of a formula for Catalan's constant that had been conjectured by Sun in 2014. Then, using a similar approach as in our first proof, we solve an open problem due to Sun involving the sequence of alternating odd harmonic numbers. Our methods, more broadly, allow us to reduce difficult binomial-harmonic sums to finite combinations of dilogarithms that are evaluable using previously known algorithms.

1. INTRODUCTION

In this article, we introduce proofs for two conjectured formulas due to Sun, namely [12] (cf. [14])

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k+1)16^k} \left(3H_{2k+1} + \frac{4}{2k+1} \right) = 8G \tag{1}$$

and [13, 14]:

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k+1)8^k} \left(\sum_{0 \leq j < k} \frac{(-1)^j}{2j+1} - \frac{(-1)^k}{2k+1} \right) = -\frac{\sqrt{2}}{16}\pi^2, \tag{2}$$

where $G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$ denotes Catalan's constant, and where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ denotes the n^{th} entry in the sequence of harmonic numbers. Prior to our proof of (2) introduced in this article, the problem of proving (2) seems to have been open.

1.1. **Preliminaries.** We refer to harmonic-type numbers of the following form as alternating odd harmonic numbers:

$$\overline{O}_n^{(m+1)} = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{(2k-1)^{m+1}} = \frac{(-1)^m}{m!} \int_0^1 \frac{1-(-1)^n x^{2n}}{1+x^2} \ln^m x \, dx. \tag{3}$$

We are to make use of the above moment formula, as below, to prove Sun's conjectured formula in (2).

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There has been much in the way of research, as of late, concerning *colored multiple zeta values* (CMZVs) [3, 4, 11, 15], and the concept of a CMZV provides us with a key tool used in our proof in Section 3 below. Following [3, 4], we write

$$\zeta(s_1, \ldots, s_k) = \sum_{s_1 > \cdots > s_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}$$

and

$$L_{s_1, \ldots, s_k}(a_1, \ldots, a_k) = \sum_{s_1 > \cdots > s_k \geq 1} \frac{a_1^{n_1} \cdots a_k^{n_k}}{n_1^{s_1} \cdots n_k^{s_k}}$$

to denote, respectively, the *multiple zeta function* and the *colored polylogarithm*, where the value k is referred to as the *length* and $s_1 + \cdots + s_k$ is referred to as the *weight*. The classical *dilogarithm function* is such that

$$\mathrm{Li}_2(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^2}. \tag{4}$$

A CMZV is an expression of the form $L_{s_1, \ldots, s_k}(a_1, \ldots, a_k)$ in the case whereby the arguments of the form a_i are N^{th} roots of unity, the indices of the form s_i are positive integers, and $(a_i, s_i) \neq (1, 1)$ for all indices i.

Catalan's constant is notable as a special value of the Dirichlet L-function

$$L(s, \chi) = \sum_{k=1}^{\infty} \frac{\chi(k)}{k^s},$$

with

$$L(s, \chi_4) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^s},$$

letting non-principal Dirichlet characters modulo n be denoted as per usual. We are to make use of values associated with

$$L(2, \chi_6) = \frac{1}{36} \left(\psi^{(1)} \left(\frac{1}{6} \right) - \psi^{(1)} \left(\frac{5}{6} \right) \right)$$

in our proof introduced in this article that Sun's conjectured formula in (1) holds true. The Euler–Mascheroni constant is such that $\gamma = \lim_{n \to \infty} \left(H_n - \ln n \right)$, and we recall that the *digamma function* refers to the special function such that the following equalities are satisfied [10, §9]:

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \sum_{n=0}^{\infty} \frac{z-1}{(n+1)(n+z)}.$$

Writing $\psi = \psi^{(0)}$, the polygamma function is such that

$$\psi^{(n)}(z) = \frac{d^n}{dz^n} \psi^{(0)}(z).$$

2. Sun's formula for G

In 2017, Ablinger [1] had mentioned that his `HarmonicSums` package may be used to prove (1), and Chen [8] had recently suggested that a finite sum identity due to Tauraso [16] could be used to prove (1), but there was no indication given in [8] as to how this could be achieved.

Theorem 2.1. *Sun's conjectured equality in* (1) *holds true (cf.* [1, 8]*).*

Proof. We are to make use of the moment formula

$$\int_0^1 (-2k-1)x^{2k}\log(1-x)\,dx = H_{2k+1} \tag{5}$$

together with the dominated convergence theorem, in the following manner. By replacing the summand factor H_{2k+1} in the series

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k+1)16^k} H_{2k+1}$$

with the definite integral in (5), and then rewriting the resultant summand so as to obtain

$$\sum_{k=0}^{\infty} \int_0^1 \frac{(-2k-1)\binom{2k}{k}x^{2k}\log(1-x)}{(2k+1)16^k}\,dx,$$

we may reverse the order of the operators $\sum_{k=0}^{\infty}\cdot$ and $\int_0^1 \cdot\,dx$ according to the dominated convergence theorem. So, by the generalized binomial theorem, we have that the equality

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}H_{2k+1}}{(2k+1)16^k} = \int_0^1 -\frac{2\log(1-x)}{\sqrt{4-x^2}}\,dx \tag{6}$$

holds true. We have determined that an antiderivative for $-\frac{2\log(1-x)}{\sqrt{4-x^2}}$ is as below, and this is easily seen by differentiating the following expression and simplifying:

$$\begin{aligned}
&-2\Bigg(i\mathrm{Li}_2\left(-\frac{2e^{i\sin^{-1}\left(\frac{x}{2}\right)}}{-i+\sqrt{3}}\right) + i\mathrm{Li}_2\left(\frac{2e^{i\sin^{-1}\left(\frac{x}{2}\right)}}{i+\sqrt{3}}\right) + \frac{1}{2}i\sin^{-1}\left(\frac{x}{2}\right)^2 \\
&-\sin^{-1}\left(\frac{x}{2}\right)\log\left(1+\frac{2e^{i\sin^{-1}\left(\frac{x}{2}\right)}}{-i+\sqrt{3}}\right) - \sin^{-1}\left(\frac{x}{2}\right)\log\left(1-\frac{2e^{i\sin^{-1}\left(\frac{x}{2}\right)}}{\sqrt{3}+i}\right) \\
&+\log(1-x)\sin^{-1}\left(\frac{x}{2}\right)\Bigg).
\end{aligned}$$

Taking limits as $x \to 0$ and $x \to 1$, this gives us that the definite integral in (6) is equal to:

$$2\left(i\text{Li}_2\left(-\frac{2}{-i+\sqrt{3}}\right)+i\text{Li}_2\left(\frac{2}{i+\sqrt{3}}\right)\right)-2i\text{Li}_2\left(-\frac{2e^{\frac{i\pi}{6}}}{-i+\sqrt{3}}\right)-\frac{13i\pi^2}{36}$$
$$+\frac{1}{3}\pi\log\left(\frac{i}{\sqrt{3}}\right)+\frac{1}{3}\pi\log\left(1-\frac{e^{\frac{i\pi}{6}}}{-\frac{\sqrt{3}}{2}+\frac{i}{2}}\right).$$

Rewriting the argument of the first dilogarithmic expression shown above as

$$\text{Li}_2\left(e^{-\frac{5i\pi}{6}}\right)$$

and then applying a series multisection to (4) according to the residue classes of the indices modulo 6, and similarly for the other dilogarithmic expressions in our evaluation of (6), this can be used to show that (6) is also equal to

$$\frac{8G}{3}-\frac{4i\pi^2}{27}+\frac{1}{9}\left(-\sqrt{3}+i\right)\psi^{(1)}\left(\frac{1}{3}\right)+\frac{1}{9}\left(\sqrt{3}+i\right)\psi^{(1)}\left(\frac{2}{3}\right),$$

and this may be confirmed with Mathematica's `FunctionExpand` command applied to our dilogarithmic form for (6). So, we have that Sun's series in (1) is expressible as

$$8G-\frac{4i\pi^2}{9}+\frac{1}{3}i\psi^{(1)}\left(\frac{1}{3}\right)-\frac{\psi^{(1)}\left(\frac{1}{3}\right)}{\sqrt{3}}+\frac{1}{3}i\psi^{(1)}\left(\frac{2}{3}\right)+\frac{\psi^{(1)}\left(\frac{2}{3}\right)}{\sqrt{3}}$$
$$+4\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)^2 16^k}.$$

The series

$$\sum_{k=0}^{\infty}\frac{16^{-k}\binom{2k}{k}}{(1+2k)^2}=1.0149416064... \tag{7}$$

is equal to a well-known mathematical constant known as *Gieseking's constant* [2], with reference to the OEIS entry `A143298` and the references therein, and it is known that Gieseking's constant is equal to

$$\frac{9-\psi^{(1)}\left(\frac{2}{3}\right)+\psi^{(1)}\left(\frac{4}{3}\right)}{4\sqrt{3}}.$$

To show that the series in (7) is equal to Gieseking's constant, we may start with the Maclaurin series for the inverse sine, so as to write

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}z^k y^{2k}}{2k+1}=\frac{\sin^{-1}\left(2y\sqrt{z}\right)}{2y\sqrt{z}},$$

and by indefinitely integrating with respect to y, we obtain

$$\frac{1}{2\sqrt{z}}\Bigg(\sin^{-1}\left(2y\sqrt{z}\right)\log\left(1-e^{2i\sin^{-1}\left(2y\sqrt{z}\right)}\right) \\ -\frac{1}{2}i\left(\sin^{-1}\left(2y\sqrt{z}\right)^2+\operatorname{Li}_2\left(e^{2i\sin^{-1}\left(2y\sqrt{z}\right)}\right)\right)\Bigg),$$

and this leads us to an expression equivalent to Gieseking's constant, using previously known dilogarithmic expressions for this constant. So, we have shown that Sun's series in (1) is equal to the following:

$$8G-\frac{4i\pi^2}{9}+\frac{1}{3}i\psi^{(1)}\left(\frac{1}{3}\right)-\frac{\psi^{(1)}\left(\frac{1}{3}\right)}{\sqrt{3}}+\frac{1}{3}i\psi^{(1)}\left(\frac{2}{3}\right)+\frac{\psi^{(1)}\left(\frac{2}{3}\right)}{\sqrt{3}} \\ +\frac{9-\psi^{(1)}\left(\frac{2}{3}\right)+\psi^{(1)}\left(\frac{4}{3}\right)}{\sqrt{3}}.$$

Since Sun's series is real-valued, we may omit the complex terms in the above expansion, giving us that Sun's series equals

$$8G+3\sqrt{3}-\frac{\psi^{(1)}\left(\frac{1}{3}\right)}{\sqrt{3}}+\frac{\psi^{(1)}\left(\frac{4}{3}\right)}{\sqrt{3}}.$$

Applying an index shift to rewrite $\psi^{(1)}\left(\frac{4}{3}\right)$ in terms of $\psi^{(1)}\left(\frac{1}{3}\right)$, using the relation such that $\psi^{(1)}(z+1)-\psi^{(1)}(z)=-\frac{1}{z^2}$, this gives us that Sun's series in (1) is reducible to $8G$, as desired. □

3. Solution to an open problem

Theorem 3.1. *Sun's conjectured equality in* (2) *holds true.*

Proof. From (3), we find that

$$\sum_{j=0}^{k-1}\frac{(-1)^j}{2j+1}=\int_0^1\frac{1-(-1)^k x^{2k}}{1+x^2}\,dx \tag{8}$$

for each natural number k. Expanding the summand in (2), we obtain the series

$$\sum_{k=0}^{\infty}\frac{\left(-\frac{1}{8}\right)^k\binom{2k}{k}}{(2k+1)^2}, \tag{9}$$

and we may obtain a dilogarithmic form for this expression in virtually exactly the same way that the closely related expression for Gieseking's constant [2] in (7) had been proved in Section 2 in order to prove Sun's formula in (1). For the sake of brevity, we refer to Section 2 for details as to how to prove that the evaluation suggested below holds, noting that Mathematica is able to obtain the following from (9) via the `FunctionExpand` command:

$$\sqrt{2}\text{Li}_2\left(-\frac{\sqrt{3}+1}{\sqrt{2}}\right)-\sqrt{2}\text{Li}_2\left(1-\frac{\sqrt{3}+1}{\sqrt{2}}\right)+\frac{\pi^2}{6\sqrt{2}}$$
$$-\frac{\log^2\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)}{\sqrt{2}}+\sqrt{2}\log\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)\log\left(1+\frac{\sqrt{3}+1}{\sqrt{2}}\right).$$

So, it remains to show that

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}\sum_{j=0}^{k-1}\frac{(-1)^j}{2j+1}}{(2k+1)8^k} \tag{10}$$

is equal to the following expression:

$$\frac{\pi^2}{24\sqrt{2}}-\frac{\log^2\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)}{\sqrt{2}}+\sqrt{2}\log\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)\log\left(1+\frac{\sqrt{3}+1}{\sqrt{2}}\right)$$
$$+\sqrt{2}\text{Li}_2\left(-\frac{\sqrt{3}+1}{\sqrt{2}}\right)-\sqrt{2}\text{Li}_2\left(1-\frac{\sqrt{3}+1}{\sqrt{2}}\right).$$

We replace the summand factor $\sum_{j=0}^{k-1}\frac{(-1)^j}{2j+1}$ in (10) with the equivalent integral in (8), and we rearrange this resultant expression so as to obtain

$$\sum_{k=0}^{\infty}\int_0^1\frac{\binom{2k}{k}}{(2k+1)8^k}\frac{1-(-1)^k x^{2k}}{1+x^2}\,dx.$$

By the dominated convergence theorem, we are allowed to switch the order of the operators $\sum_{k=0}^{\infty}\cdot$ and $\int_0^1\cdot\,dx$, giving us

$$\int_0^1\frac{\pi x-4\sinh^{-1}\left(\frac{x}{\sqrt{2}}\right)}{2\sqrt{2}x\left(x^2+1\right)}\,dx,$$

according to the generating function given by the Maclaurin series for $\sinh^{-1}$. We have determined the followed antiderivative for the above integrand, and this may be verified by differentiating the following expression and then simplifying:

$$\frac{1}{2\sqrt{2}}\Bigg(2\mathrm{Li}_2\left(-\frac{(1+i)e^{\sinh^{-1}\left(\frac{x}{\sqrt{2}}\right)}}{\sqrt{2}}\right)+2\mathrm{Li}_2\left(-\frac{(1-i)e^{\sinh^{-1}\left(\frac{x}{\sqrt{2}}\right)}}{\sqrt{2}}\right)$$
$$+2\mathrm{Li}_2\left(\frac{(1-i)e^{\sinh^{-1}\left(\frac{x}{\sqrt{2}}\right)}}{\sqrt{2}}\right)+2\mathrm{Li}_2\left(\frac{(1+i)e^{\sinh^{-1}\left(\frac{x}{\sqrt{2}}\right)}}{\sqrt{2}}\right)$$
$$-2\mathrm{Li}_2\left(e^{2\sinh^{-1}\left(\frac{x}{\sqrt{2}}\right)}\right)+\pi\tan^{-1}(x)+2\sinh^{-1}\left(\frac{x}{\sqrt{2}}\right)$$
$$\log\left(1-\frac{(1+i)e^{\sinh^{-1}\left(\frac{x}{\sqrt{2}}\right)}}{\sqrt{2}}\right)$$
$$+2\sinh^{-1}\left(\frac{x}{\sqrt{2}}\right)\log\left(1-\frac{(1-i)e^{\sinh^{-1}\left(\frac{x}{\sqrt{2}}\right)}}{\sqrt{2}}\right)$$
$$+2\sinh^{-1}\left(\frac{x}{\sqrt{2}}\right)\log\left(1+\frac{(1-i)e^{\sinh^{-1}\left(\frac{x}{\sqrt{2}}\right)}}{\sqrt{2}}\right)$$
$$+2\sinh^{-1}\left(\frac{x}{\sqrt{2}}\right)\log\left(1+\frac{(1+i)e^{\sinh^{-1}\left(\frac{x}{\sqrt{2}}\right)}}{\sqrt{2}}\right)$$
$$-4\sinh^{-1}\left(\frac{x}{\sqrt{2}}\right)\log\left(1-e^{2\sinh^{-1}\left(\frac{x}{\sqrt{2}}\right)}\right)\Bigg).$$

Setting $x \to 1$ and $x \to 0$ and then taking the difference, and then subtracting the dilogarithmic form for (10), it remains to prove that the following expression vanishes:

$$\frac{1}{48\sqrt{2}}\Bigg(12\Bigg(\mathrm{Li}_2\left(-7-4\sqrt{3}\right)-8\mathrm{Li}_2\left(-\sqrt{2+\sqrt{3}}\right)-4\mathrm{Li}_2\left(2+\sqrt{3}\right)$$
$$+8\mathrm{Li}_2\left(1-\sqrt{2+\sqrt{3}}\right)+\log^2\left(2-\sqrt{3}\right)+4\log\left(2-\sqrt{3}\right)\log\left(1+\sqrt{2+\sqrt{3}}\right)$$
$$+\log(16)\operatorname{csch}^{-1}\left(\sqrt{2}\right)\Bigg)+13\pi^2-96i\pi\operatorname{csch}^{-1}\left(\sqrt{2}\right)\Bigg).$$

According to the elementary dilogarithmn identity

$$\mathrm{Li}_2(x)+\mathrm{Li}_2(1-x)=\frac{1}{6}\pi^2-\log x\log(1-x),$$

it remains to prove that the following expression vanishes:

$$\frac{1}{48\sqrt{2}}\Bigg(29\pi^2 - 48i\pi\left(2\operatorname{csch}^{-1}\left(\sqrt{2}\right) + \log\left(2+\sqrt{3}\right)\right)$$
$$+ 12\Bigg(\operatorname{csch}^{-1}\left(\sqrt{2}\right)\log(16)$$
$$+ \log^2\left(2-\sqrt{3}\right) - 4\log\left(2+\sqrt{3}\right)\log\left(-1+\sqrt{2+\sqrt{3}}\right)$$
$$+ 4\log\left(2-\sqrt{3}\right)\log\left(1+\sqrt{2+\sqrt{3}}\right) + \mathrm{Li}_2\left(-7-4\sqrt{3}\right) - 8\mathrm{Li}_2\left(-\sqrt{2+\sqrt{3}}\right)$$
$$- 8\mathrm{Li}_2\left(\sqrt{2+\sqrt{3}}\right) - 4\mathrm{Li}_2\left(2+\sqrt{3}\right)\Bigg)\Bigg).$$

Using the elementary dilogarithm identity

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(-x) = \frac{1}{2}\mathrm{Li}_2(x^2),$$

it remains to prove that the following expression vanishes:

$$\frac{29\pi^2}{48\sqrt{2}} - i\sqrt{2}\pi\operatorname{csch}^{-1}\left(\sqrt{2}\right) + \frac{\operatorname{csch}^{-1}\left(\sqrt{2}\right)\log(16)}{4\sqrt{2}} + \frac{\log^2\left(2-\sqrt{3}\right)}{4\sqrt{2}}$$
$$- \frac{i\pi\log\left(2+\sqrt{3}\right)}{\sqrt{2}} - \frac{\log\left(2+\sqrt{3}\right)\log\left(-1+\sqrt{2+\sqrt{3}}\right)}{\sqrt{2}}$$
$$+ \frac{\log\left(2-\sqrt{3}\right)\log\left(1+\sqrt{2+\sqrt{3}}\right)}{\sqrt{2}} + \frac{\mathrm{Li}_2\left(-7-4\sqrt{3}\right)}{4\sqrt{2}} - \sqrt{2}\mathrm{Li}_2\left(2+\sqrt{3}\right).$$

So, it remains to prove the closed form for

$$\frac{1}{8}\mathrm{Li}_2\left(-\left(2+\sqrt{3}\right)^2\right) - \mathrm{Li}_2\left(2+\sqrt{3}\right)$$

suggested by the purportedly vanishing expression indicated above. Using Landen's identity, this is equivalent to

$$-\frac{1}{16}\log^2\left(1+\left(2+\sqrt{3}\right)^2\right) - \frac{1}{8}\mathrm{Li}_2\left(\frac{2+\sqrt{3}}{4}\right) - \mathrm{Li}_2\left(2+\sqrt{3}\right).$$

The two-term linear combination

$$\frac{1}{8}\mathrm{Li}_2\left(\frac{2+\sqrt{3}}{4}\right) + \mathrm{Li}_2\left(2+\sqrt{3}\right) \tag{11}$$

of dilogarithmic values is a CMZV of level 12 and weight 2 [6]. CMZVs of this form are completely tabulated via the Mathematica package concerning multiple zeta values due to Au [5, 6]. After loading the required package [5, 6], inputting

```
MZPolyLog[{0, 1}, (2 + Sqrt[3])/4]
```

provides the desired evaluation, which holds according to the algorithms corresponding to the package `MultipleZetaValues` [5, 6]. □

4. Discussion

Our derivation of the two-term dilogarithm relation for (11) is quite experimental, as there are many "black boxes" involved in this derivation and the underlying algorithms [6], in something of an analogous way compared to the famous Wilf–Zeilberger method [9]. We provide, as below, a more complete explanation as to how the closed form (11) may be obtained via the algorithms we had applied.

The required package contains a basis B for a $\mathbb{Q}$-vector space of level 12, weight 2 CMZVs. We denote this $\mathbb{Q}$-space as V [6]. The basis B may be accessed via the following command [6].

```
MZBasis[12,2]
```

After inputting

```
MZPolyLog[{0, 1}, (2 + Sqrt[3])/4]
```

Au's package determines that the above expression lies in V, and an algorithm is applied to determine how this expression may be written in terms of the members of B, noting that $\mathrm{Li}_2(2-\sqrt{3})$ is in B, and that we may rewrite the desired two-term Li_2 identity in an equivalent way so that the following holds [6]:

$$\mathrm{Li}_2\left(\frac{\sqrt{3}+2}{4}\right) - 8\mathrm{Li}_2(2-\sqrt{3}) = -\frac{\pi^2}{4} - 2\log^2(2) + \frac{5}{2}\log^2(\sqrt{3}+2) - 2\log(\sqrt{3}+2)\log(2).$$

The foregoing considerations inspire a full exploration as to how our proof in the preceding Section may be generalized, by reducing the series in Sun's conjectures to finite combinations of polylogarithms of "reasonable" weight, and then invoking Au's algorithms. On the other hand, our two-term dilogarithm relations as above are of interest in their own right, in view of recent results as in [7].

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