On the recursion relation of Motzkin numbers of higher rank

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Abstract

It is proposed that finding the recursion relation and generating function for the (colored) Motzkin numbers of higher rank introduced recently is an interesting problem.

1 Introduction

The classical Motzkin numbers count the numbers of Motzkin paths (and are also related to many other combinatorial objects [1]). Let us recall the definition of Motzkin paths. We consider in the Cartesian plane $\mathbf{Z} \times \mathbf{Z}$ those lattice paths starting from $(0,0)$ that use the steps $\{U, L, D\}$, where $U = (1,1)$ is an up-step, $L = (1,0)$ a level-step and $D = (1,-1)$ a down-step. Let $M(n,k)$ denote the set of paths beginning in $(0,0)$ and ending in (n,k) that never go below the x-axis. Paths in $M(n,0)$ are called *Motzkin paths* and $m_n := |M(n,0)|$ is called *n-th Motzkin number*. The Motzkin numbers satisfy the recursion relation [2]

$$(n+2)m_n = (2n+1)m_{n-1} + 3(n-1)m_{n-2} \tag{1}$$

and have the generating function [1]

$$\sum_{n \geq 0} m_n x^n = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}. \tag{2}$$

Those Motzkin paths which have no level-steps are called *Dyck paths* and are enumerated by Catalan numbers [1]. In recent times the above situation has been generalized by introducing colorings of the paths. For example, the *k-colored Motzkin paths* have horizontal steps colored by k colors (see [3, 4] and the references given therein). More generally, one can introduce colors for each type of step [5, 6]. Let us denote by u the number of colors for an up-step U, by l the number of colors for a level-step L and by d the number of colors for a down-step D. (Note that if we normalize the weights as $u + l + d = 1$ we can view the paths as discrete random walks.) One can then introduce the set $M^{(u,l,d)}(n,0)$ of (u,l,d)-*colored Motzkin paths* and the corresponding (u,l,d)-*Motzkin numbers* $m_n^{(u,l,d)} := |M^{(u,l,d)}(n,0)|$. In [5] a combinatorial proof is given that the $(1,l,d)$-Motzkin numbers satisfy the recursion relation

$$(n+2)m_n^{(1,l,d)} = l(2n+1)m_{n-1}^{(1,l,d)} + (4d - l^2)(n-1)m_{n-2}^{(1,l,d)}. \tag{3}$$

Choosing $l = 1$ and $d = 1$ yields the recursion relation (1) of the conventional Motzkin numbers $m_n \equiv m_n^{(1,1,1)}$. Note that choosing $(u,l,d) = (1,k,1)$ corresponds to the k-colored

Motzkin paths. Defining $m_{k,n} := |M^{(1,k,1)}(n,0)|$, one obtains from (3) the recursion relation $(n+2)m_{k,n} = k(2n+1)m_{k,n-1} + (4-k^2)(n-1)m_{k,n-2}$ for the number of k-colored Motzkin paths. A generating function for $m_{k,n}$ is derived in [3],

$$\sum_{n\geq 0} m_{k,n}x^n = \frac{1-kx-\sqrt{(1-kx)^2-4x^2}}{2x^2}. \tag{4}$$

For $k=1$ this identity reduces to (2) for the conventional Motzkin numbers $m_n \equiv m_{1,n}$.

Problem 1. *Derive a recursion relation and generating function for the general* (u,l,d)*-Motzkin numbers* $m_n^{(u,l,d)}$*, i.e., generalize (3) and (4) to the general case.*

2 Motzkin numbers of higher rank

We will now generalize the situation considered in the previous section. It is discussed in [7] in the context of duality triads of higher rank (where one considers recurrence relations of higher rank, or equivalently, orthogonal matrix polynomials [8]) why it is interesting to consider the situation where the steps of the paths can go up or down more than one unit. The maximum number of units which a single step can go up or down will be called the rank. More precisely, let $r \geq 1$ be a natural number. The set of *admissable* steps consists of:

1. r types of up-steps $U_j = (1,j)$ with weights u_j for $1 \leq j \leq r$.
2. A level-step $L = (1,0)$ with weight l.
3. r types of down-steps $D_j = (1,-j)$ with weights d_j for $1 \leq j \leq r$.

In the following we write $(\mathbf{u}, l, \mathbf{d}) := (u_r, \ldots, u_1, l, d_1, \ldots, d_r)$ for the vector of weights.

Definition 1. [7] The set $M^{(\mathbf{u},l,\mathbf{d})}(n,0)$ of $(\mathbf{u},l,\mathbf{d})$*-colored Motzkin paths of rank* r is the set of paths which start in $(0,0)$, end in $(n,0)$, have only admissable steps and are never below the x-axis. The corresponding number of paths, $m_n^{(\mathbf{u},l,\mathbf{d})} := |M^{(\mathbf{u},l,\mathbf{d})}(n,0)|$, will be called $(\mathbf{u},l,\mathbf{d})$*-Motzkin number of rank* r.

Remark 1. The case $r=1$ corresponds exactly to the (u,l,d)-Motzkin paths (and numbers) considered in the previous section. In the case of higher rank one may also switch to a probabilistic point of view if one considers the normalization $u_r + \cdots + u_1 + l + d_1 + \cdots + d_r = 1$. Furthermore, in close analogy to the rank one case one may also define *Dyck paths of rank* r as those Motzkin paths of rank r which have no level-steps.

It is clear that we can associate to each Motzkin path of rank r and length n a conventional Motzkin path of length rn in the following fashion (for the following we assume all weights to be equal to one). For each admissable step $S_k \in \{U_j, L, D_j\}$ we let $\mu(U_j) := U^j \equiv UU\cdots U$ (j times), $\mu(L) := L$ and $\mu(D_j) := D^j$. For a path $P_n = S_1 S_2 \cdots S_n$ we define $\mu(P_n)$ by concatenation, i.e., $\mu(P_n) := \mu(S_1)\mu(S_2)\cdots\mu(S_n)$. For example, if $r=3$ and $P_4 = U_3 L D_2 D_1$, then $\mu(P_4) = UUULDDD$ is a path of length 7. To obtain a path of length $3 \cdot 4 = 12$, we fill the missing 5 steps with L's. More formally, let us introduce the *absolute height* $|S_k|$ of a step S_k by $|U_j| := j, |L| := 0$ and $|D_j| := j$. The absolute height of a path $P_n = S_1 \cdots S_n$

is given as the sum of the absolute heights of its steps, i.e., $|P_n| = \sum_{k=1}^{n} |S_k|$. With this notation we have well-defined maps

$$\begin{aligned} \mu_{r,n} : M^{(\mathbf{1},1,\mathbf{1})}(n,0) &\longrightarrow M(rn,0), \\ P_n &\longmapsto \mu_{r,n}(P_n) := \mu(P_n)L^{rn-|P_n|}. \end{aligned}$$

The map $\mu_{r,n}$ is in general neither surjective nor injective. As an example, consider $r = 2$ and $n = 2$. $M^{(\mathbf{1},1,\mathbf{1})}(2,0) = \{LL, U_1D_1, U_2D_2\}$. It follows that $\mu_{2,2}(LL) = LLLL, \mu_{2,2}(U_1D_1) = UDLL$ and $\mu_{2,2}(U_2D_2) = UUDD$ are in $M(4,0)$ but there are many more elements in $M(4,0)$ which are not in the image of $M^{(\mathbf{1},1,\mathbf{1})}(2,0)$, e.g., the path $ULLD$. This shows that $\mu_{2,2}$ is not surjective. On the other hand, consider $r = 2$ and $n = 3$. The two paths $U_1U_2D_3$ and $U_2U_1D_3$ in $M^{(\mathbf{1},1,\mathbf{1})}(3,0)$ have the same image $UUUDDD$ in $M(6,0)$, i.e., $\mu_{2,3}$ is not injective. This brief discussion should show that the study of $M^{(\mathbf{u},l,\mathbf{d})}(n,0)$, i.e., the case of higher rank, cannot be reduced in a straightforward way to the rank one case.

3 The Problem

Problem 2. *Derive a recursion relation and generating function for $m_n^{(\mathbf{u},l,\mathbf{d})}$, the general $(\mathbf{u},l,\mathbf{d})$-Motzkin numbers of rank r.*

Remark 2. Clearly, Problem 2 captures Problem 1 as the case $r = 1$. Presumably, the most interesting case should be the first case where r is greater than one, i.e., $r = 2$, since already here many of the arguments used in [5] break down. A very simple example of such an argument in the case $r = 1$ is that a path ending in $(n,0)$, i.e., on the x-axis, must have an equal number of up- and down-steps (implying in particular that there do not exist Dyck paths of length n if n is odd). This is not true in the case of higher rank since already for $r = 2$ one can find a Motzkin path (even Dyck path) $U_2D_1D_1$ of length 3 with unequal number of up- and down-steps.

Motzkin paths are related to duality triads of rank one, whereas Motzkin paths of rank r are related to duality triads of rank r (see [7] for a discussion of duality triads and their relation to Motzkin numbers and [8] for some further properties of duality triads). Duality triads of rank r are characterised by a recursion relation of order $2r+1$. This is the reason for the following conjecture.

Conjecture 1. *The $m_n^{(\mathbf{u},l,\mathbf{d})}$ satisfy a $(2r+1)$-term recursion relation.*

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