On the Signed Small Ball Inequality

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Abstract

Let h_R denote an L^∞ normalized Haar function adapted to a dyadic rectangle $R \subset [0,1]^d$. We show that for all choices of coefficients $\alpha(R) \in \{\pm 1\}$, we have the following lower bound on the L^∞ norms of the sums of such functions, where the sum is over rectangles of a fixed volume.

$$n^{\eta(d)} \lesssim \Big\| \sum_{|R|=2^{-n}} \alpha(R) h_R(x) \Big\|_{L^\infty([0,1]^d)}, \quad \text{for all} \quad \eta(d) < \frac{d-1}{2} + \frac{1}{8d},$$

where the implied constant is independent of $n \geq 1$. The inequality above (without restriction on the coefficients) arises in connection to several areas, such as Probabilities, Approximation, and Discrepancy. With $\eta(d) = (d-1)/2$, the inequality above follows from orthogonality, while it is conjectured that the inequality holds with $\eta(d) = d/2$. This is known and proved in (Talagrand, 1994) in the case of $d = 2$, and recent papers of the of the authors (Bilyk and Lacey, 2006), (Bilyk et al., 2007) prove that in higher dimensions one can take $\eta(d) > (d-1)/2$, without specifying a particular value of η. The restriction $\alpha_R \in \{\pm 1\}$ allows us to significantly simplify our prior arguments and to find an explicit value of $\eta(d)$.

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1 The Small Ball Conjectures

In one dimension, the class of dyadic intervals are $\mathcal{D} := \{[j2^k, (j+1)2^k) : j,k \in \mathbb{Z}\}$. Each dyadic interval has a left and right half, which are also dyadic. Define the L^∞-normalized Haar functions

$$h_I := -\mathbf{1}_{I_{\text{left}}} + \mathbf{1}_{I_{\text{right}}}.$$

In d dimensions, a *dyadic rectangle* is a product of dyadic intervals, i.e. an element of $\mathcal{D}^d$. We define a Haar function associated to R to be the product of the Haar functions associated with each side of R, namely

$$h_{R_1 \times \cdots \times R_d}(x_1, \ldots, x_d) := \prod_{j=1}^{d} h_{R_j}(x_j).$$

We will consider a local problem and concentrate on rectangles with fixed volume. This is the 'hyperbolic' assumption, that pervades the subject. Our concern is the following Theorem and Conjecture concerning a *lower bound* on the L^∞ norm of sums of hyperbolic Haar functions:

Small Ball Conjecture 1.1. *For dimension $d \geq 3$ we have the inequality*

$$2^{-n} \sum_{|R|=2^{-n}} |\alpha(R)| \lesssim n^{\frac{1}{2}(d-2)} \Big\| \sum_{|R|\geq 2^{-n}} \alpha(R) h_R \Big\|_\infty \tag{1.2}$$

Average case analysis — that is passing through L^2 — shows that we always have

$$2^{-n} \sum_{|R|=2^{-n}} |\alpha(R)| \lesssim n^{\frac{1}{2}(d-1)} \Big\| \sum_{|R|\geq 2^{-n}} \alpha(R) h_R \Big\|_\infty$$

Namely, the constant on the right is bigger than in the conjecture by a factor of $\sqrt{n}$. We refer to this as the 'average case estimate,' and refer to improvements over this as a 'gain over the average case estimate.'

Random choices of coefficients $\alpha(R)$ show that the Small Ball Conjecture is sharp. The interest in this conjecture arises from questions in Probability Theory (Talagrand, 1994), Approximation Theory (Temlyakov, 1989) and the theory of Irregularities of Distribution (Beck and Chen, 1987).

In dimension $d = 2$, the Conjecture was resolved by (Talagrand, 1994).[1]

Talagrand's Theorem 1.3. *In dimension $d = 2$, we have*

$$2^{-n} \sum_{|R|=2^{-n}} |\alpha(R)| \lesssim \Big\| \sum_{|R|\geq 2^{-n}} \alpha(R) h_R \Big\|_\infty \tag{1.4}$$

Here, the sum on the right is taken over all rectangles with area at least 2^{-n}.

In dimensions $d \geq 3$, there is partial information in (Bilyk and Lacey, 2006), (Bilyk et al., 2007), which builds upon the method devised by (Beck, 1989).

Theorem 1.5. *In dimension $d \geq 4$, there is a $\zeta(d) > 0$ so that for all choices of coefficients $\alpha(R)$ we have*

$$2^{-n} \sum_{|R|=2^{-n}} |\alpha(R)| \lesssim n^{\frac{d-1}{2} - \zeta(d)} \Big\| \sum_{|R|\geq 2^{-n}} \alpha(R) h_R \Big\|_\infty \tag{1.6}$$

The Conjecture 1.1 appears to be quite difficult to resolve in dimensions $d \geq 3$, and in this paper we discuss a more restrictive formulation of the conjecture that still appears to be of interest.

Signed Small Ball Conjecture 1.7. *We have the inequality (1.2), in the case where the coefficients $\alpha(R) \in \{\pm 1\}$, for $|R| = 2^{-n}$. Namely, under these assumptions on the coefficients $\alpha(R)$ we have the inequality*

$$n^{d/2} \lesssim \Big\| \sum_{|R|=2^{-n}} \alpha(R) h_R \Big\|_\infty . \tag{1.8}$$

The main result of this note is the next Theorem, in which we give an explicit gain over the trivial bound in the Signed Small Ball Conjecture in dimensions $d \geq 3$.

Theorem 1.9. *In dimension $d \geq 3$, for choices of coefficients $\alpha(R) \in \{\pm 1\}$, we have the inequality*

$$n^{\eta(d)} \lesssim \Big\| \sum_{|R|=2^{-n}} \alpha(R) h_R \Big\|_\infty , \quad \text{for all} \quad \eta(d) < \frac{d-1}{2} + \frac{1}{8d} . \tag{1.10}$$

The interest in the Theorem above is that the amount of the gain is explicit, and that the method of proof, using essential ingredients from (Bilyk et al., 2007; Bilyk and Lacey, 2006; Beck, 1989), is much simpler than either of these prior works.

The principal difficulty in three and higher dimensions is that two dyadic rectangles of the same volume can share a common side length. Beck (Beck, 1989) found a specific estimate in this case, an estimate that is extended in (Bilyk et al., 2007; Bilyk and Lacey, 2006). The reader is encouraged to consult (Bilyk et al., 2007) for a more detailed exposition of the methods and applications. The main simplification in the current note lies in the equalities (3.9), which allow us to avoid analyzing longer coincidences. The value of η appears to be the optimal one we can get out of this line of reasoning, imputing additional interest to the methods of proof used to improve this estimate.

[1] This result should be compared to (Schmidt, 1972), as well as (Temlyakov, 1995).

2 Notations and Littlewood-Paley Inequality

Let $\vec{r} \in \mathbb{N}^d$ be a partition of n, thus $\vec{r} = (r_1, \dots, r_d)$, where the r_j are nonnegative integers and $|\vec{r}| := \sum_t r_t = n$, which we refer to as the *length of the vector* $\vec{r}$. Denote all such vectors as $\mathbb{H}_n$. ('$\mathbb{H}$' for 'hyperbolic.') For vector $\vec{r}$ let $\mathcal{R}_{\vec{r}}$ be all dyadic rectangles R such that for each coordinate k, $|R_k| = 2^{-r_k}$.

Definition 2.1. We call a function f an r *function with parameter* $\vec{r}$ if

$$f = \sum_{R \in \mathcal{R}_{\vec{r}}} \beta_R \, h_R \,, \qquad \beta_R \in \{\pm 1\} \,. \tag{2.2}$$

A fact used without further comment is that $f_{\vec{r}}^2 \equiv 1$.

It is an essential remark that the fact that the β_R above are arbitrary forces a good deal of the complexity in the conjectures listed in this paper. The r functions we are interested in are:

$$f_{\vec{r}} := \sum_{R \in \mathcal{R}_{\vec{r}}} \alpha(R) \, h_R \tag{2.3}$$

An important distinction between the signed and general cases is that the function whose L^∞ norm we are trying to estimate is a sum of r functions.

We recall those Littlewood-Paley inequalities of most interest to us. Notice that due to the L^∞ normalization some of the equalities here will look odd to a reader accustomed to the L^2 normalization.

Littlewood-Paley Inequalities 2.4. *In one dimension, we have the inequalities*

$$\Big\| \sum_{I \subset \mathbb{R}} a_I h_I(\cdot) \Big\|_p \lesssim \sqrt{p} \Big\| \Big[\sum_{I \subset \mathbb{R}} a_I^2 \mathbf{1}_I(\cdot) \Big]^{1/2} \Big\|_p \,, \qquad 2 < p < \infty \,. \tag{2.5}$$

Moreover, these inequalities continue to hold in the case where the coefficients a_I take values in a Hilbert space $\mathcal{H}$.

The growth of the constant is essential for us, in particular the factor $\sqrt{p}$ is, up to a constant, the best possible in this inequality. See (Fefferman and Pipher, 1997; Wang, 1991). That these inequalities hold for Hilbert space valued sums is imperative for applications to higher dimensional sums of Haar functions. The relevant inequality is as follows.

Theorem 2.6. *We have the inequalities below for hyperbolic sums of* r *functions in dimension* $d \geq 3$.

$$\Big\| \sum_{|\vec{r}|=n} f_{\vec{r}} \Big\|_p \lesssim (pn)^{(d-1)/2} \,, \qquad 2 < p < \infty \,. \tag{2.7}$$

3 Proof of Theorem 1.9

The proof of the Theorem is by duality, namely we construct a function Ψ of L^1 norm about one, which is used to provide a lower bound on the L^∞ norm of the sum of Haar functions.

The function Ψ will take the form of a Riesz product, but in order to construct it, we need some definitions first. Fix $0 < \kappa < 1$, with the interesting choices of κ being close to zero. Define relevant parameters by

$$q = \lfloor a n^{\varepsilon} \rfloor \,, \qquad \varepsilon = \frac{1}{2d} - \kappa \,, \qquad b = \tfrac{1}{4} \,, \tag{3.1}$$

$$\widetilde{\rho} = a q^b n^{-(d-1)/2} \,, \qquad \rho = \sqrt{q} n^{-(d-1)/2} . \tag{3.2}$$

Here a is a small positive constant, we use the notation $b = \frac{1}{4}$ throughout, so as not to obscure those aspects of the argument that that dictate this choice of parameter. $\widetilde{\rho}$ is a 'false' L^2 normalization for the sums we consider, while the larger term ρ is the 'true' L^2 normalization. Our 'gain over the average case estimate' in the Small Ball Conjecture is $q^b \simeq n^{\varepsilon/4} = n^{1/8d - \kappa/4} = n^{\eta(d) - (d-1)/2}$.

Divide the integers $\{1, 2, \dots, n\}$ into q disjoint increasing intervals of equal length $I_1, \dots, I_q$, and let $\mathbb{A}_t := \{\vec{r} \in \mathbb{H}_n \,:\, r_1 \in I_t\}$. Let

$$F_t := \sum_{\vec{r} \in \mathbb{A}_t} f_{\vec{r}} \,, \qquad H := \sum_{\vec{r} \in \mathbb{H}_n} f_{\vec{r}} = \sum_{t=1}^{q} F_t \,. \tag{3.3}$$

The Riesz product is a 'short product.'

$$\Psi := \prod_{t=1}^{q} (1 + \widetilde{\rho} F_t)\,, \qquad \Psi_{\neq j} := \prod_{\substack{t=1 \\ t\neq j}}^{q} (1 + \widetilde{\rho} F_t)\,, \quad 1 \le j \le q\,.$$

Note the subtle way that the false L^2 normalization enters into the product. It means that the product is, with high probability, positive. And of course, for a positive function F, we have $\mathbb{E}F = \|F\|_1$, with expectations being typically easier to estimate. This heuristic is made precise below. Notice also that $\mathbb{E}\Psi = 1$.

We need a final bit of notation. Set

$$\Phi_t := \sum_{\substack{\vec{r}\neq\vec{s}\in\mathbb{A}_t \\ r_1=s_1}} f_{\vec{r}} \cdot f_{\vec{s}}\,. \tag{3.4}$$

Note that in this sum, there are $2d-3$ free parameters among the vectors $\vec{r}$ and $\vec{s}$. That is, the pair of vectors $(\vec{r},\vec{s})$ are completely specified by there values in $2d-3$ coordinates.

Our main Lemma is below. Note that in (3.8), the assertion is that the $2d-3$ parameters in the definition of Φ_t behave, with respect to L^p norms, as if they are independent.

Lemma 3.5. *We have these estimates.*

$$\|\Psi\|_1 \lesssim 1\,, \tag{3.6}$$

$$\|\Psi\|_2\,,\ \max_{1\le j\le n} \|\Psi_{\neq j}\|_2 \lesssim e^{a' q^{2b}}\,, \tag{3.7}$$

$$\|\Phi_t\|_p \lesssim p^{d-1/2} n^{d-3/2} q^{-1/2}\,, \qquad 2 < p < \infty\,. \tag{3.8}$$

In (3.7), the value of a' is a decreasing function of $0 < a < 1$.

The proof of this Lemma is taken up next section. Assuming the Lemma, we proceed as follows. An important simplification in the Signed Small Ball Inequality comes from the equalities

$$\begin{aligned} \langle F_j, \Psi\rangle &= \Big\langle \sum_{\vec{r}\in\mathbb{A}_j} f_{\vec{r}}, \Psi \Big\rangle \\ &= \sum_{\vec{r}\in\mathbb{A}_j} \langle f_{\vec{r}}, (1+\widetilde{\rho} F_j)\Psi_{\neq j}\rangle \\ &= \widetilde{\rho} \sum_{\vec{r}\in\mathbb{A}_j} \langle f_{\vec{r}}^2, \Psi_{\neq j}\rangle + \widetilde{\rho}\langle \Phi_j, \Psi_{\neq j}\rangle \\ &= \widetilde{\rho}\,\sharp\mathbb{A}_j + \widetilde{\rho}\langle \Phi_j, \Psi_{\neq j}\rangle\,. \end{aligned} \tag{3.9}$$

We have used the fact that there has to be a coincidence in the first coordinate in order for the product of r functions to have non-zero integral. The first term in the third line is the 'diagonal' term, while the second term arises from different vectors which coincide in the first coordinate. Therefore, we can estimate

$$\begin{aligned} \|H\|_\infty &\gtrsim \langle H, \Psi\rangle \\ &= \sum_{j=1}^{q} \langle F_j, \Psi\rangle \\ &= \widetilde{\rho}\,\sharp\mathbb{H}_n + \sum_{j=1}^{q} \widetilde{\rho}\langle \Phi_j, \Psi_{\neq j}\rangle \end{aligned}$$

It is clear that

$$\widetilde{\rho}\,\sharp\mathbb{H}_n \simeq a^{5/4} n^{\frac{d-1}{2}+\frac{\varepsilon}{4}} = a^{5/4} n^{\eta(d)}\,, \tag{3.10}$$

which is our principal estimate. The other term we treat as an error term. Using Hölder's inequality, and (3.6) and (3.7) we see that

$$\|\Psi_{\neq j}\|_{(q^{2b})'} \le \|\Psi_{\neq j}\|_1^{(q^{2b}-2)/q^{2b}} \|\Psi_{\neq j}\|_2^{2q^{-2b}} \lesssim 1\,.$$

Therefore, we can estimate as below, where we use the estimate above and (3.8).

$$\Big| \sum_{j=1}^{q} \widetilde{\rho}\langle \Phi_j, \Psi_{\neq j}\rangle \Big| \lesssim \sum_{j=1}^{q} \widetilde{\rho} \|\Phi_j\|_{q^{2b}} \|\Psi_{\neq j}\|_{(q^{2b})'}$$

$$\lesssim q \cdot \frac{aq^b}{n^{(d-1)/2}} \cdot q^{2b(d-1/2)} n^{d-3/2} \cdot q^{-1/2}$$
$$\simeq aq^{2bd+1/2} n^{(d-2)/2} \ll n^{\eta(d)} .$$

This term will be smaller than the term in (3.10). The proof of our main result is complete, modulo the proof of Lemma 3.5.

4 The Analysis of the Coincidence and Corollaries of the Beck Gain

Following the language of J. Beck (Beck, 1989), a *coincidence* occurs if we have two vectors $\vec{r} \neq \vec{s}$ with e. g. $r_1 = s_1$, precisely the condition that we imposed in the definition of Φ_t, (3.4). He observed that sums over products of r functions in which there are coincidences obey favorable L^2 estimates. We refer to (extensions of) this observation as the *Beck Gain.*

The Simplest Instance of the Beck Gain 4.1. *We have the estimates below, valid for an absolute implied constant that is only a function of dimension $d \geq 3$.*

$$\sup_{1 \leq j \leq n} \|\Phi_j\|_p \lesssim p^{d-1/2} \cdot n^{d-3/2} q^{-1/2} , \qquad 1 \leq p < \infty . \tag{4.2}$$

This Lemma, in dimension $d = 3$ appears in (Bilyk and Lacey, 2006). The proof in higher dimensions, which was given in (Bilyk et al., 2007), is inductive. We omit the proof here as it is rather lengthy and refer the reader to (Bilyk et al., 2007) for the details. The estimate in the aforementioned papers does not strictly speaking contain Lemma 4.1, as it does not include $q^{-1/2}$. However, this can be easily fixed in the proof due to the fact that the value of the first coordinate can be chosen in n/q ways rather than n. We also emphasize that the estimate above may admit an improvement, in that the power of p is perhaps too large by a single power.

Conjecture 4.3. *We have the estimates below, valid for an absolute implied constant that is only a function of dimension $d \geq 3$.*

$$\sup_{1 \leq j \leq n} \|\Phi_j\|_p \lesssim (pn)^{d-3/2} q^{-1/2} , \qquad 1 \leq p < \infty . \tag{4.4}$$

With this conjecture we could prove our main theorem for all $\eta(d) < \frac{d-1}{2} + \frac{1}{8d-8}$.

The Beck Gain Lemma 4.1 has several important consequences. Theorem 2.6 implies an exponential estimate for sums of r functions. However, with the Beck Gain at hand, we can derive a subgaussian estimate for such sums, for moderate deviations.

Theorem 4.5. *Using the notation of (3.2) and (3.3), we have this estimate, valid for all $1 \leq t \leq q$.*

$$\|\rho F_t\|_p \lesssim \sqrt{p} , \qquad 1 \leq p \leq c \left(\frac{n}{q}\right)^{\frac{1}{2d-1}} . \tag{4.6}$$

As a consequence, we have the distributional estimate

$$\mathbb{P}(|\rho F_t| > x) \lesssim \exp(-cx^2) , \qquad x < c \left(\frac{n}{q}\right)^{\frac{1}{4d-2}} . \tag{4.7}$$

Here $0 < c < 1$ is an absolute constant.

Proof. Recall that

$$F_t = \sum_{\vec{r} \in \mathbb{A}_t} f_{\vec{r}} .$$

where $\mathbb{A}_t := \{\vec{r} \in \mathbb{H}_n \,:\, r_1 \in I_t\}$, and I_t in an interval of integers of length n/q, so that $\sharp \mathbb{A}_t \simeq n^2/q \simeq \rho^{-2}$.

Apply the Littlewood-Paley inequality in the first coordinate. This results in the estimate

$$\begin{aligned} \|\rho F_t\|_p &\lesssim \sqrt{p} \Big\| \Big[\sum_{s \in I_j} \Big| \rho \sum_{\vec{r} \,:\, r_1 = s} f_{\vec{r}} \Big|^2 \Big]^{1/2} \Big\|_p \\ &\lesssim \sqrt{p} \|1 + \rho^2 \Phi_t\|_{p/2}^{1/2} \\ &\lesssim \sqrt{p} \Big\{ 1 + \|\rho^2 \Phi_t\|_{p/2}^{1/2} \Big\} . \end{aligned}$$

Here, it is important to use the constants in the Littlewood-Paley inequalities that give the correct order of growth of $\sqrt{p}$. Of course the terms Φ_t are controlled by the estimate in (4.4). In particular, we have

$$\|\rho^2\Phi_t\|_p \lesssim \frac{q}{n^{d-1}} p^{d-1/2} n^{d-3/2} q^{-1/2} \lesssim p^{d-1/2} n^{-1/2} q^{1/2}\,. \tag{4.8}$$

Hence (4.6) follows.

The second distributional inequality is a well known consequence of the norm inequality. Namely, one has the inequality below, valid for all x:

$$\mathbb{P}(\rho F_t > x) \le C^p p^{p/2} x^{-p}\,, \qquad 1 \le p \le c\left(\frac{n}{q}\right)^{\frac{1}{2d-1}}\,.$$

If x is as in (4.7), we can take $p \simeq x^2$ to prove the claimed exponential squared bound. □

Proof of (3.6). Observe that

$$\mathbb{P}(\Psi < 0) \lesssim q \exp(ca^{-2}q^{1-2b})\,.$$

Indeed, using (4.7), we have

$$\begin{aligned}\mathbb{P}(\Psi < 0) &\le \sum_{t=1}^{q} \mathbb{P}(\widetilde{\rho} F_t < -1)\\ &= \sum_{t=1}^{q} \mathbb{P}(\rho F_t < -a^{-1}q^{1/2-b})\\ &\lesssim q \exp(ca^{-2}q^{1-2b})\,.\end{aligned}$$

Note that to be able to use (4.7) we need to have $a^{-1}q^{1/2-b} \le c\left(\frac{n}{q}\right)^{\frac{1}{2d-1}}$, which leads to $\varepsilon < \frac{1}{2d}$. Then, assuming (3.7), we have

$$\begin{aligned}\|\Psi\|_1 &= \mathbb{E}\Psi - 2\mathbb{E}\Psi \mathbf{1}_{\Psi<0}\\ &\le 1 + 2\mathbb{P}(\Psi < 0)^{1/2}\|\Psi\|_2\\ &\lesssim 1 + \exp(-a^{-2}q^{1-2b}/2 + aq^{2b})\,.\end{aligned}$$

For sufficiently small $0 < a < 1$, the proof is finished. Note that this last step forces $b = 1/4$ on us.

□

Proof of (3.7). The supremum over j will be an immediate consequence of the proof below, and so we don't address it specifically.

Let us give the initial, essential observation. We expand

$$\mathbb{E}\prod_{j=1}^{q}(1+\widetilde{\rho}F_j)^2 = \mathbb{E}\prod_{j=1}^{q}(1+2\widetilde{\rho}F_j + (\widetilde{\rho}F_j)^2)\,.$$

fixed, and let $\mathcal{F}$ be the sigma field generated by $F_1, \dots, F_{q-1}$. We have

$$\begin{aligned}\mathbb{E}\Big(1 + 2\widetilde{\rho}F_q + (\widetilde{\rho}F_q)^2 \,\Big|\, \mathcal{F}\Big) &= 1 + \mathbb{E}\Big((\widetilde{\rho}F_q)^2 \,\Big|\, \mathcal{F}\Big)\\ &= 1 + a^2 q^{2b-1} + \widetilde{\rho}^2\Phi_q\,,\end{aligned} \tag{4.9}$$

where Φ_q is defined in (3.4). Then, we see that

$$\begin{aligned}\mathbb{E}\prod_{v=1}^{q}(1+2\widetilde{\rho}F_t + (\widetilde{\rho}F_t)^2) &= \mathbb{E}\Big\{\prod_{v=1}^{q-1}(1+2\widetilde{\rho}F_t + (\widetilde{\rho}F_t)^2) \times \mathbb{E}\Big(1 + 2\widetilde{\rho}F_t + (\widetilde{\rho}F_t)^2 \,\Big|\, \mathcal{F}\Big)\Big\}\\ &\le (1 + a^2q^{2b-1})\mathbb{E}\prod_{v=1}^{q-1}(1+2\widetilde{\rho}F_t + (\widetilde{\rho}F_t)^2) \qquad (4.10)\\ &\quad + \mathbb{E}|\widetilde{\rho}^2\Phi_q| \cdot \prod_{v=1}^{q-1}(1+2\widetilde{\rho}F_t + (\widetilde{\rho}F_t)^2) \qquad (4.11)\end{aligned}$$

This is the main observation: one should induct on (4.10), while treating the term in (4.11) as an error, as the 'Beck Gain' estimate (4.4) applies to it.

Let us set up notation to implement this line of approach. Set

$$N(V;r) := \Big\| \prod_{v=1}^{V} (1 + \widetilde{\rho} F_v) \Big\|_r , \qquad V = 1, \dots, q .$$

We will obtain a very crude estimate for these numbers for $r = 4$. Fortunately, this is relatively easy for us to obtain. Namely, q is small enough that we can use the inequalities (4.6) to see that

$$\begin{aligned} N(V;4) &\le \prod_{v=1}^{V} \|1 + \widetilde{\rho} F_v\|_{4V} \\ &\le (1 + Cq^b)^V \\ &\le (Cq)^q . \end{aligned}$$

For a large choice of $\tau > 1$, which is a function of the choice of $\kappa > 0$ in (3.1), we have the estimate below from Hölder's inequality

$$N(V; 2(1 - 1/\tau q)^{-1}) \le N(V;2)^{1-2/\tau q} \cdot N(V;4)^{2/\tau q} . \tag{4.12}$$

We see that (4.10), (4.11) and (4.12) give us the inequality

$$\begin{aligned} N(V+1;2)^2 &\le (1 + a^2 q^{2b-1}) N(V;2)^2 + C \cdot N(V; 2(1-1/\tau q)^{-1})^2 \cdot \|\widetilde{\rho}^2 \Phi_V\|_{\tau q} \\ &\le (1 + a^2 q^{2b-1}) N(V;2)^2 + C N(V;2)^{2-4/\tau q} \cdot N(V;4)^{4/\tau q} \|\widetilde{\rho}^2 \Phi_V\|_{\tau q} \\ &\le (1 + a^2 q^{2b-1}) N(V;2)^2 + C_\tau q^{d-1/2+4/\tau} n^{-1/2} N(V;2)^{2-2/\tau q} . \end{aligned} \tag{4.13}$$

In the last line we have used the the inequality (4.4) and the constant C_τ is only a function of $\tau > 1$, which is fixed.

Of course we only apply this as long as $N(V;2) \ge 1$. Assuming this is true for all $V \ge 1$, we see that

$$N(V+1;2)^2 \le (1 + a^2 q^{2b-1} + C_\tau q^{d-1/2+4/\tau} n^{-1/2}) N(V;2)^2 .$$

And so, by induction,

$$N(q;2) \lesssim (1 + a^2 q^{2b-1} + C_\tau q^{d-1/2+4/\tau} n^{-1/2})^{q/2} \lesssim \mathrm{e}^{2aq^{2b}} .$$

Here, the last inequality will be true for large n, provided τ is much bigger than $1/\kappa$. Indeed, we need

$$a^2 q^{2b-1} \ge C_\tau q^{d-1/2+4/\tau} n^{-1/2}$$

Or equivalently,

$$c n^{1/2} \ge q^{d+4/\tau} .$$

Comparing to the definition of q in (3.1), we see that the proof is finished.

□

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