The discrepancy of a needle on a checkerboard

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Abstract

Consider the plane as a checkerboard, with each unit square colored black or white in an arbitrary manner. We show that for any such coloring there are straight line segments, of arbitrarily large length, such that the difference of their white length minus their black length, in absolute value, is at least the square root of their length, up to a multiplicative constant. For the corresponding "finite" problem ($N \times N$ checkerboard) we also prove that we can color it in such a way that the above quantity is at most $C\sqrt{N \log N}$, for any placement of the line segment.

1 Introduction

In this paper we answer a question posed to us by P. Papasoglu [3]:

> Consider the plane as a checkerboard, with each unit square
>
> $$[m, m+1) \times [n, n+1), \quad m, n \in \mathbb{Z},$$
>
> colored black or white. Is it possible that there is such a coloring and a finite constant M such that for any line segment I placed on the checkerboard the difference of its white length minus its black length is, in absolute value, at most M?

We show below that the answer is negative.

Theorem 1. *Suppose each unit cell $[i, i+1) \times [j, j+1)$, $i, j \in \mathbb{Z}$, in the plane is painted black or white. Then there exist arbitrarily long line segments I such that the difference of the black part of I minus its white part is at least $C\sqrt{|I|}$ in absolute value, where $C > 0$ is an absolute constant.*

Our approach is Fourier analytic and Theorem 1 follows in an obvious way from the "finite" Theorem 2 below, by setting $z_{ij} = 1$ if the cell $[i, i+1) \times [j, j+1)$ is black and $z_{ij} = -1$ if the cell is white.

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Theorem 2. *Suppose the function $f : \mathbb{R}^2 \to \mathbb{C}$ is zero outside the square $[0, N] \times [0, N]$ and is constant and equal to z_{ij} on each of the squares $(i, j) + [0, 1)^2$, $0 \le i, j < N$. Then there is a straight line S such that*

$$\left| \int_S f \right| \ge C N^{-1/2} \left(\sum_{i,j=0}^{N-1} |z_{ij}|^2 \right)^{1/2},$$

where C is a positive constant.

It is possible to give an almost matching upper bound for Theorem 2. In Theorem 3 we show how to color a $N \times N$ checkerboard so that all line segments placed on the checkerboard have white-over-black excess (discrepancy) bounded by $C\sqrt{N \log N}$.

Theorem 3. *For each N there exists a function $f : \mathbb{R}^2 \to \{0, \pm 1\}$ which is zero outside the square $[0, N] \times [0, N]$ and is constant and equal to 1 or -1 on each of the squares $(i, j) + [0, 1)^2$, $0 \le i, j < N$, and is such that on any straight line segment I we have*

$$\left| \int_I f \right| \le C\sqrt{N \log N}. \tag{1}$$

Unfortunately this does not translate to a coloring of the infinite checkerboard so that the discrepancy of any line segment I is $o(|I|)$. This problem we leave open.

The question dealt with in this paper falls naturally into the subject of geometric discrepancy [2]. In this research area there is usually an underlying measure μ as well as a family $\mathcal{F}$ of allowed subsets of Euclidean space, on which the measure μ is evaluated and upper and lower bounds are sought on the range of μ on $\mathcal{F}$. The most classical case is that where μ is a normalized collection of points masses in the unit square minus Lebesgue measure and $\mathcal{F}$ consists of all axis-aligned rectangles in the unit square. Usually the underlying measure μ has an atomic part (point masses) and the family $\mathcal{F}$ consists of "fat" sets. In the problem we are studying here the measure μ has no atomic part (it is absolutely continuous) and the collection $\mathcal{F}$ consists of all straight line segments, which may be considered thin sets, and, strictly speaking, μ is 0 on these sets. We believe that the closest work in the bibliography to this paper is that of Rogers [4] where the measure μ is the same as here but the family $\mathcal{F}$ consists not of straight line segments but of thin strips. There does not seem to exist a connection of our results with those of [4].

2 Proofs

The definition of the Fourier Transform that we use is

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi \cdot x} \, dx$$

for $f \in L^1(\mathbb{R}^d)$ and $\xi \in \mathbb{R}^d$.

We use the letter C in this paper as an absolute positive constant, not the same in all its occurrences.

Proof. (of Theorem 2)
For a straight line L through the origin let us denote by $\pi_L f$ (the projection of f

onto L) the function of $t \in \mathbb{R}$ given by

$$\pi_L f(t) = \int_{\mathbb{R}} f(tu + su^{\perp})\, ds,$$

where u is a unit vector along L and $u^{\perp}$ is a unit vector orthogonal to u. It is well known and easy to see that the Fourier Transform of $\pi_L f$ is equal to the restriction of the (two-dimensional) Fourier Transform of f, $\widehat{f}$, on L:

$$\widehat{\pi_L f}(\xi) = \widehat{f}(\xi u), \quad \xi \in \mathbb{R}.$$

Write $M = \sup_{L,t} |\pi_L f(t)|$ where the supremum is taken over all lines L through the origin and real numbers t. We aim to prove that

$$M > CN^{-1/2} \left(\sum_{i,j=0}^{N-1} |z_{ij}|^2 \right)^{1/2},$$

for some constant $C > 0$.

Since the support of f has diameter $\leq CN$ it follows from Parseval's equality that

$$\int_{\mathbb{R}} \left| \widehat{f}(tu) \right|^2 dt = \int_{\mathbb{R}} |\pi_L f(t)|^2\, dt \leq CM^2 N. \tag{2}$$

It also follows from Parseval's equality that

$$\sum_{i,j=0}^{N-1} |z_{ij}|^2 = \int_{\mathbb{R}^2} |f|^2 = \int_{\mathbb{R}^2} \left| \widehat{f} \right|^2. \tag{3}$$

Lemma 1. *For sufficiently large $A > 0$ we have*

$$\int_{|\xi|>A} \left| \widehat{f}(\xi) \right|^2 d\xi \leq \frac{C}{A} \sum_{i,j=0}^{N-1} |z_{ij}|^2.$$

Proof. If $Q = [0,1]^2$ then by a simple calculation

$$\widehat{\chi_Q}(\xi_1, \xi_2) = e^{2\pi i(\xi_1+\xi_2)/2} \frac{\sin \pi\xi_1}{\pi\xi_1} \frac{\sin \pi\xi_2}{\pi\xi_2}$$

and, therefore, with $\xi = (\xi_1, \xi_2)$ we have

$$|\widehat{\chi_Q}(\xi)|^2 \leq \frac{C}{(1+|\xi_1|)^2(1+|\xi_2|)^2}.$$

If $G = \{(m,n) : 0 \leq m, n < N\}$ then

$$f = \chi_Q * \sum_{p \in G} z_p \delta_p,$$

where δ_p denotes the unit point mass at p and z_p is the value of f in the cell $p + [0,1)^2$. It follows that $\widehat{f}(\xi) = \widehat{\chi_Q}(\xi)\phi(\xi)$, where the trigonometric polynomial $\phi(\xi)$ is given by

$$\phi(\xi) = \sum_{p \in G} z_p e^{2\pi i p \cdot \xi}.$$

The function ϕ is $\mathbb{Z}^2$-periodic and $\int_Q |\phi|^2 = \sum_{i,j=0}^{N-1} |z_{ij}|^2$. We have

$$\begin{aligned}
\int_{|\xi|>A} \left|\widehat{f}(\xi)\right|^2 d\xi &\leq \int_{|\xi_1|>0.1A \text{ or } |\xi_2|>0.1A} \left|\widehat{f}(\xi)\right|^2 d\xi \\
&\leq \int_{|\xi_1|>0.1A \text{ or } |\xi_2|>0.1A} \frac{C|\phi(\xi_1,\xi_2)|^2}{(1+|\xi_1|)^2(1+|\xi_2|)^2}\, d\xi \\
&\leq \sum_{m>0.1A, \text{ or } n>0.1A} \frac{C}{(1+|m|)^2(1+|n|)^2} \int_{Q+(m,n)} |\phi|^2 \\
&= \int_Q |\phi|^2 \cdot \sum_{m>0.1A, \text{ or } n>0.1A} \frac{C}{(1+|m|)^2(1+|n|)^2} \\
&\leq \sum_{i,j=0}^{N-1} |z_{ij}|^2 \cdot \frac{C}{A},
\end{aligned}$$

which proves the lemma. □

It follows from Lemma 1 and (3) that, for sufficiently large $A > 0$, we have

$$\frac{1}{2} \sum_{i,j=0}^{N-1} |z_{ij}|^2 < \int_{|\xi|<A} \left|\widehat{f}(\xi)\right|^2$$

Rewriting the right hand side in polar coordinates and using (2) we get

$$\begin{aligned}
\frac{1}{2} \sum_{i,j=0}^{N-1} |z_{ij}|^2 &\leq A \int_{u\in S^1} \int_{|t|<A} \left|\widehat{f}(tu)\right|^2 \\
&\leq CAM^2N,
\end{aligned}$$

which proves the desired inequality. □

Proof. (of Theorem 3)
The proof is probabilistic. We choose the value of f in each cell (i,j), $0 \leq i,j < N$, to be ± 1 with equal probability and independently of other cells.

Fix a line segment I which is contained in $[0,N] \times [0,N]$ and let $\ell_1, \ldots, \ell_k$ be the lengths of the intersections of I with each of the cells it intersects. Then $\int_I f = \sum_{j=1}^k \epsilon_j \ell_j$, where $\epsilon = \pm 1$ independently and uniformly.

We use the standard estimate (see for instance [1, Appendix A]):

$$\mathbf{Pr}\left[\left|\int_I f\right| > \lambda\sigma\right] \leq Ce^{-C\lambda^2}$$

where $\sigma^2 = \sum_{j=1}^k \ell_j^2$. Since the ℓ_j are bounded by $\sqrt{2}$ it follows that

$$\sigma \leq C\sqrt{\sum_{j=1}^k \ell_j} \leq C\sqrt{|I|} \leq C\sqrt{N},$$

hence, using $\lambda = C\sqrt{\log N}$ we obtain

$$\mathbf{Pr}\left[\left|\int_I f\right| \geq C\sqrt{N \log N}\right] \leq CN^{-K}, \tag{4}$$

where we can choose the positive number K as large as we please, by choosing the appropriate constant C in the left hand side of (4).

Our next step is to apply the bound (4) to a well chosen set $\mathcal{S}$ of line segments such that the validity of (1) for the elements of $\mathcal{S}$ implies the validity of (1) for all segments (but with a larger constant). It is important that $|\mathcal{S}|$ must not grow faster than a fixed power of N, which will allow us to use (4) simultaneously for all segments in $\mathcal{S}$.

A straight line segment I is determined by its endpoints, call them a_I and b_I. Our set $\mathcal{S}$ will contain all segments that are determined by a grid of points of spacing $N^{-10} \times N^{-10}$. We will show that for any line segment I there is a line segment $J \in \mathcal{S}$ such that

$$\left| \int_I f - \int_J f \right| \leq 1. \tag{5}$$

First, we assume that the line segment I is not nearly horizontal or nearly vertical. That is we assume that the line segment I forms an angle between N^{-1} and $\frac{\pi}{2} - N^{-1}$ with the x-axis. In this case it is obvious that if the endpoints of J are $O(N^{-10})$ apart from those of I then, for any cell K that I intersects at length ℓ, the intersection of J with K has length $\ell + O(N^{-5})$, from which (5) clearly follows.

Let us now deal with the case of I being nearly horizontal. (The case of nearly vertical segments is treated similarly.) In this case the segment I is either entirely contained in one horizontal strip

$$[0, N] \times [i, i+1] \tag{6}$$

or is contained in two successive such strips. The latter case can be treated by breaking up I into two segments each of which is contained in one strip, so we assume that I is contained in strip (6). Take then J to be a member of $\mathcal{S}$ whose endpoints are $O(N^{-10})$ apart from those of I and which is also contained in strip (6), and (5) follows.

Finally, by an application of (4) with $K = 100$, we conclude, that with positive probability, for all $I \in \mathcal{S}$ we have $\left| \int_I f \right| \leq C\sqrt{N \log N}$. Hence, the latter inequality is true with high probability for all line segments I. □

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