Avoiding permutation patterns of type $(2,1)$ in compositions

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Abstract

We classify compositions avoiding a single permutation pattern of type $(2,1)$ according to Wilf-equivalence and give the generating function for each of the Wilf classes.

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1 Introduction

Pattern avoidance was first studied for $\mathfrak{S}_n$, the set of permutations of $[n] = \{1, 2, \ldots, n\}$, avoiding a pattern $\tau \in \mathfrak{S}_3$. Knuth [7] found that for any $\tau \in \mathfrak{S}_3$, the number of permutations of $[n]$ avoiding τ is given by the nth Catalan number. Later, Simion and Schmidt [11] determined $|\mathfrak{S}_n(T)|$, the number of permutations of $[n]$ simultaneously avoiding any given set of patterns $T \subseteq \mathfrak{S}_3$. Burstein [1] extended this to words of length n on the alphabet $[k] = \{1, \ldots, k\}$, determining the number of words that avoid a set of patterns $T \subseteq \mathfrak{S}_3$. Burstein and Mansour [2] extended to forbidden patterns with repeated letters.

Recently, pattern avoidance has been studied for compositions. Heubach and Mansour [4] counted the number of times a subword pattern τ of length 2 occurs in compositions, and determined the number of compositions avoiding such a pattern. They also investigated 3-letter subword patterns [6], and Mansour and Sirhan [8] considered specific ℓ-letter subword patterns in compositions. Savage and Wilf [9] considered (classical) pattern avoidance in compositions for a single pattern $\tau \in \mathfrak{S}_3$, and showed that the number of compositions of n with parts in $\mathbb{N}$ avoiding $\tau \in \mathfrak{S}_3$ is independent of τ. Savage and Wilf posed some open questions, one of which asked about (classical) pattern avoidance in compositions for patterns with repeated letters. Heubach and Mansour [5] answered this question for all such patterns of length 3, and determined the Wilf classes for avoidance of pairs of (classical) multi-patterns of length 3 in compositions. They gave generating functions for all but one class, and also considered some patterns of arbitrary length.

In this paper we focus on generalized patterns of length 3, those that have some adjacency requirements. (Classical patterns have no adjacency requirements, while subword patterns require all parts to be adjacent.) The only patterns of length 3 that have partial adjacency requirements are those that require two letters to be adjacent. We will give a complete characterization of these patterns in terms of their Wilf-equivalence and derive the generating functions for each of the different classes. We start by defining our notation in Section 2. Sections 3 and 4 contain the main results: first the classification into Wilf-equivalence classes, then the corresponding generating functions for each class.

2 Preliminaries

Let $\mathbb{N}$ be the set of all positive integers, and let A be any ordered finite (or infinite) set of positive integers, say $A = \{a_1, a_2, \ldots, a_d\}$, where $a_1 < a_2 < a_3 < \cdots < a_d$. For ease of notation, "ordered set" will always refer to a set whose elements are listed in increasing order. We use the notation A_j to denote the subset of the first j elements of A, i.e., $A_j = \{a_1, \ldots, a_j\}$ and $A = A_d$.

A *composition* $\sigma = \sigma_1\sigma_2 \ldots \sigma_m$ of $n \in \mathbb{N}$ is an ordered collection of one or more positive integers whose sum is n. The number of *summands* or *letters*, namely m, is called the number of *parts* of the composition. For any ordered set $A = \{a_1, a_2, \ldots, a_k\} \subseteq \mathbb{N}$, we denote the set of all compositions of n with parts in A (with m parts in A) by $\mathcal{C}_n^A$ ($\mathcal{C}_{n;m}^A$). We say that the composition $\sigma \in \mathcal{C}_{n;m}^A$ *contains* a permutation pattern $\tau = ab\text{-}c$ of type $(1, 2)$ if there exist i, j such that $2 \leq i + 1 < j \leq m$ and $\sigma_i\sigma_{i+1}\sigma_j$ is a subsequence isomorphic to abc, where $abc \in \mathfrak{S}_3$. Otherwise, we say that σ *avoids* τ and write $\sigma \in \mathcal{AC}_n^A(\tau)$ ($\mathcal{AC}_{n;m}^A(\tau)$). Since all patterns in this paper are permutation patterns of type $(2, 1)$, we will refer to them just as patterns.

For a given a pattern τ and an ordered finite or infinite set A of positive integers, we define $|\mathcal{AC}_{n;0}^A(\tau)| = 1$ for all $n \geq 0$ and $|\mathcal{AC}_{n;m}^A(\tau)| = 0$ for $n < 0$ or $m < 0$. We define the generating function for the number of τ-avoiding compositions of n with m parts in A as

$$AC_A^\tau(x, y) = \sum_{n,m\geq 0} |\mathcal{AC}_{n;m}^A(\tau)| x^n y^m,$$

and denote the corresponding generating function for those compositions that start with $\sigma_1\sigma_2 \ldots \sigma_k$ by $AC_A^\tau(\sigma_1\sigma_2 \ldots \sigma_k|x, y)$. Finally, we say that two patterns τ and τ' belong to the same *cardinality* or *Wilf class*, or are *Wilf-equivalent*, if for all values of A, m and n, we have $|\mathcal{AC}_{n;m}^A(\tau)| = |\mathcal{AC}_{n;m}^A(\tau')|$. In this case, we write $\tau \sim \tau'$.

3 Wilf-equivalence for type $(2, 1)$ permutation patterns

We first determine the Wilf-equivalence classes for permutation patterns of type $(2, 1)$. There are six such patterns, namely 12-3, 13-2, 21-3, 23-1, 31-2 and 32-1. These patterns fall into three separate equivalence classes. Not surprisingly, the classes split according to the last part of the pattern. One might expect that a simply reversing those parts of the compositions that correspond to the adjacent pair would do the trick of showing Wilf-equivalence. This is indeed the case for two of the equivalence classes, but does not work for showing that 13-2 $\sim$ 31-2. We will start with the easy case.

Theorem 3.1 *For any ordered set $A = \{a_1, a_2, \ldots\} \subseteq \mathbb{N}$, 12-3 $\sim$ 21-3 and 23-1 $\sim$ 32-1.*

Proof. We give a bijection ϕ between the set of compositions of n with m parts in A avoiding the respective patterns. Let $\sigma \in \mathcal{AC}^A_{n;m}(12\text{-}3)$ and assume that σ has maximal part a_j which occurs s times. Thus, σ can be decomposed as

$$\sigma^{(1)} a_j \sigma^{(2)} a_j \cdots a_j \sigma^{(s)} a_j \sigma',$$

where each $\sigma^{(i)}$ is a non-increasing composition with parts in A_{j-1} and σ' is a composition with parts in A_{j-1} that avoids 12-3. We define $\phi(\sigma)$ recursively as

$$R(\sigma^{(1)}) a_j R(\sigma^{(2)}) a_j \cdots a_j R(\sigma^{(s)}) a_j \phi(\sigma'),$$

where R is the reversal map defined by $R\colon \sigma_1\sigma_2\cdots\sigma_m \mapsto \sigma_m \cdots \sigma_2\sigma_1$. Clearly, σ avoids 12-3 if and only if $\phi(\sigma)$ avoids 21-3 and σ and $\phi(\sigma)$ are both compositions of n with m parts in A. Thus, 12-3 $\sim$ 21-3; the proof for 23-1 $\sim$ 32-1 follows with appropriate adjustments. $\square$

Now we deal with the harder equivalence.

Theorem 3.2 *For any ordered set $A = \{a_1, a_2, \ldots\} \subseteq \mathbb{N}$, 13-2 $\sim$ 31-2.*

Proof.

We define an algorithm that transforms $\sigma \in \mathcal{AC}^A_{n;m}(13\text{-}2)$ into $\sigma' \in \mathcal{AC}^A_{n;m}(31\text{-}2)$ and vice versa, thereby giving a bijection between $\mathcal{AC}^A_{n;m}(13\text{-}2)$ and $\mathcal{AC}^A_{n;m}(31\text{-}2)$. The basic idea is to move blocks of "1"s from one side of the (single) "3" to the other, leaving the corresponding (single)"2" in place. This process transforms a 13-2 pattern into a 31-2 pattern and vice versa. We make this idea precise with the following definitions: An *ascent* in σ is an integer σ_i such that $\sigma_i < \sigma_{i+1}$. The ascent σ_i is called *active* if there is an integer σ_j such that $i+1 < j$ and $\sigma_i < \sigma_j < \sigma_{i+1}$, i.e., an active ascent is the "1" in an occurrence of the pattern 13-2 of *width* $j - i + 1$. Note that an active ascent can be part of more than one occurrence of 13-2 and $\sigma \in \mathcal{AC}^A_{n;m}(13\text{-}2)$ cannot have an active ascent. For each occurrence of a pattern 13-2 we define the associated *ascent block* to be the maximal substring $\sigma_k\sigma_{k+1}\cdots\sigma_i$ such that $\sigma_\ell < \sigma_j$ for $\ell = k, \ldots, i$. Similarly, a *descent* in a composition σ is an integer $\sigma_i, i > 1$, such that $\sigma_{i-1} > \sigma_i$. The descent σ_i is called *active* if there is an integer σ_j such that $i < j$ and $\sigma_i < \sigma_j < \sigma_{i-1}$, i.e., an active descent is the "1" in an occurrence of the pattern 31-2 of *width* $j - i$. As before, an active descent can belong to more than one occurrence of 31-2, and $\sigma \in \mathcal{AC}^A_{n;m}(31\text{-}2)$ cannot have an active descent. For each occurrence of a pattern 31-2 we define the associated *descent block* to be the maximal substring $\sigma_i\sigma_{i+1}\cdots\sigma_k$ such that $\sigma_\ell < \sigma_j$ for $\ell = i, \ldots, k$. Both the ascent and the descent block consist of all "1"s. For a fixed ascent/descent block, the *innermost pattern* is the associated one of smallest width, while the *outermost pattern* is the associated pattern of largest width.

We now describe the map $\rho\colon \mathcal{AC}^A_{n;m}(13\text{-}2) \mapsto \mathcal{AC}^A_{n;m}(31\text{-}2)$. Let $\sigma = \sigma_1 \cdots \sigma_m \in \mathcal{AC}^A_{n;m}(13\text{-}2)$ have r occurrences of the pattern 13-2. Let $\sigma^{(0)} = \sigma$ and $\sigma^{(j)}$ be the composition that results after j steps of the algorithm. Basically, each step transforms one of the active descents and removes at least one of its associated occurrences of the pattern 31-2, so that after at most r

steps we obtain a composition $\rho(\sigma) \in \mathcal{AC}^A_{n;m}(31\text{-}2)$. Note that if $r = 0$, then $\sigma \in \mathcal{AC}^A_{n;m}(31\text{-}2)$ and $\rho(\sigma) = \sigma^{(0)} = \sigma$. Now assume that $r > 0$. Then $\sigma^{(j)}$ is obtained from $\sigma^{(j-1)}$ as follows: Let σ_{d_j} be the leftmost active descent in $\sigma^{(j-1)}$, and for $i = 1, \ldots, m$, let σ_i denote the ith part in $\sigma^{(j-1)}$. For the active descent σ_{d_j} identify the associated innermost 31-2 pattern. Assume that it occurs at $\sigma_{d_j-1}\sigma_{d_j}\sigma_{j*}$. Since it has the smallest width, the descent block consists of $\sigma_{d_j} \cdots \sigma_{j*-1}$. Furthermore, since σ avoids 13-2, we have that $\sigma_\ell \le \sigma_{j*-1} \le \sigma_{d_j}$ or $\sigma_\ell \ge \sigma_{j*}$ for $\ell > j*$. Now we cut out the descent block and move it to the left of σ_{d_j}, inserting it immediately after the rightmost part σ_{i*} with $\sigma_{i*} \le \sigma_{j*}$, or at the beginning of $\sigma^{(j-1)}$ if such a σ_{i*} does not exist. This insertion may create a descent if $\sigma_{i*} > \sigma_{d_j}$, but the newly created descent cannot be active due to the definition of $i*$ and the consequences of the 13-2 avoidance. We have therefore reduced the number of 31-2 patterns by at least one. Let the resulting composition be $\sigma^{(j)}$. Note that the movement of the descent block for the innermost pattern modifies other occurrences of 31-2 patterns associated with the active descent. Sometimes several patterns are removed at once; if not, then the part that previously played the role of the "2" for one of the associated patterns is now playing the role of the "1", and thus may become a new active descent. However, it occurs to the right of the previous active descent, and the set of values which can play the role of "2" for this potential active descent has decreased. Therefore, after at most r applications of the algorithm, all occurrences of 31-2 have been removed from σ, and the resulting composition $\rho(\sigma)$ is in $\mathcal{AC}^A_{n;m}(31\text{-}2)$. In addition, $\rho(\sigma)$ has at least one active ascent (created from the active descent in the last step). The resulting composition $\rho(\sigma)$ is unique, and if $\sigma \neq \tilde{\sigma}$, then $\rho(\sigma) \neq \rho(\tilde{\sigma})$. This gives $|\mathcal{AC}^A_{n;m}(13\text{-}2)| \ge |\mathcal{AC}^A_{n;m}(31\text{-}2)|$.

To compute the image $\rho'(\sigma)$ of $\sigma \in \mathcal{AC}^A_{n;m}(31\text{-}2)$, modify the algorithm for ρ accordingly: in the jth step identify the rightmost active ascent and its associated outermost 13-2 pattern. Assume that this 13-2 pattern occurs at $\sigma_{d_j}\sigma_{d_j+1}\sigma_{j*}$. Insert its ascent block immediately before σ_{j*}. Again, the resulting composition $\rho'(\sigma)$ is unique, and if $\sigma \neq \tilde{\sigma}$, then $\rho'(\sigma) \neq \rho'(\tilde{\sigma})$. This gives $|\mathcal{AC}^A_{n;m}(31\text{-}2)| \ge |\mathcal{AC}^A_{n;m}(13\text{-}2)|$, and therefore, the two sets have the same number of compositions. □

We give a few examples to illustrate the two algorithms. Note that in each case, $\rho'(\rho(\sigma)) = \sigma$, even though the intermediate compositions are not necessarily the same. In addition, the number of patterns associated with active descents/ascents do not have to be the same in the composition and its image, and not even the number of active ascents and descent have to be the same.

Example 3.3 *Let* $\sigma = 59424511241 \in \mathcal{AC}^{[9]}_{38;11}(13\text{-}2)$. *Note that* σ *has two active descents with associated* 31-2 *patterns* 945, 512, *and* 514. *It is transformed as follows:*

$$59424511241 \to 54249511241 \to 54211495241 \to 54211429541 \in \mathcal{AC}^{[9]}_{38;11}(31\text{-}2),$$

corresponding to the movements of descent blocks (424) *inserted after* 5, (11) *inserted after* 2, *and* (2) *inserted after* 4. *On the other hand, starting with* $\sigma = 54211429541 \in \mathcal{AC}^{[9]}_{38;11}(31\text{-}2)$ *(having two active ascents with associated* 13-2 *patterns* 142, 295, *and* 294*) we obtain*

$$54211429541 \to 54211495241 \to 59421142541 \to 59424511241 \in \mathcal{AC}_{38;11^{[9]}}(13\text{-}2),$$

corresponding to the movements of ascent blocks (2) *inserted before* 4, (42114) *inserted before* 5, *and* (11) *inserted before* 4.

As a second example, we consider $\sigma = 9445421126718 \in \mathcal{AC}^{[9]}_{54;13}(13\text{-}2)$ *with one active descent and associated* 31-2 *patterns* 945*,* 946*,* 947*,* 948*. This composition is transformed as follows:*

$$9445421126718 \to 4495421126718 \to 4454211296718 \to 4454211269718 \to 4454211267198,$$

corresponding to movement of the blocks (44) *inserted before 9,* (542112) *inserted after 4,* (6) *inserted after 2, and* (71) *inserted after 6. The resulting composition* $4454211267198 \in \mathcal{AC}^{[9]}_{54;13}(31\text{-}2)$ *has one active ascent, but only a single associated* 13-2 *pattern, namely* 198*. The reverse map therefore has only one intermediate step, where the block* (44542112671) *is inserted before the 8:*

$$4454211267198 \to 9445421126718 \in \mathcal{AC}^{[9]}_{54;13}(13\text{-}2).$$

Finally, we give an example where the image has fewer active ascents. Let

$$\sigma = 6244582418191 \in \mathcal{AC}^{[9]}_{55;13}(13\text{-}2),$$

which has two active descents with associated 31-2 *patterns* 624*,* 624*,* 625*, and* 824*. The image is created as follows:*

$$6244582418191 \to 2644582418191 \to 2446582418191 \to 2442658418191 \in \mathcal{AC}^{[9]}_{55;13}(31\text{-}2),$$

corresponding to the movements of descent blocks (2) *inserted before* 6*,* (44) *inserted after* 2*, and* (2) *inserted after* 4*. The resulting composition has only one active ascent with two associated patterns* 265 *and* 264*. The reverse map is given by*

$$2442658418191 \to 2446582418191 \to 6244582418191 \in \mathcal{AC}^{[9]}_{55;13}(13\text{-}2),$$

corresponding to the movements of ascent blocks (2) *inserted before* 4 *and* (244) *inserted before* 5*.*

So altogether we have that 12-3 $\sim$ 21-3, 23-1 $\sim$ 32-1 and 13-2 $\sim$ 31-2. In fact these are all the Wilf classes for patterns of type $(2,1)$, since the sequences for the number of compositions of n that avoid the respective patterns are different (see Examples 4.3, 4.6 and 4.9).

4 Generating functions for type $(2,1)$ permutation patterns

In order to present our next result we need the following lemma.

Lemma 4.1 *For all* $d \geq 0$ *we have*

$$1 + \sum_{i=1}^{d} \frac{x^i y}{\prod_{j=1}^{i}(1 - x^j y)} = \frac{1}{\prod_{j=1}^{d}(1 - x^j y)}.$$

Proof. We proceed the proof by induction on d. Clearly, the lemma holds for $d=0$ and $d=1$. Assume that the lemma holds for $d-1$. Then for $d \geq 2$ we have that

$$\begin{aligned} 1+\sum_{i=1}^{d}\frac{x^iy}{\prod_{j=1}^{i}(1-x^jy)} &= 1+\sum_{i=1}^{d-1}\frac{x^iy}{\prod_{j=1}^{i}(1-x^jy)}+\frac{x^dy}{\prod_{j=1}^{d}(1-x^jy)} \\ &= \frac{1}{\prod_{j=1}^{d-1}(1-x^jy)}+\frac{x^dy}{\prod_{j=1}^{d}(1-x^jy)} \qquad \text{by induction hypothesis} \\ &= \frac{1-x^dy}{\prod_{j=1}^{d}(1-x^jy)}+\frac{x^dy}{\prod_{j=1}^{d}(1-x^jy)} \\ &= \frac{1}{\prod_{j=1}^{d}(1-x^jy)}. \end{aligned}$$

Hence, by the principle of induction the desired identity is true for all $d \geq 0$. □

We now derive the generating functions for the set $A=[d]$.

Theorem 4.2 *The generating function for the number of compositions of n with m parts in $[d]$ that avoid 12-3 is given by*

$$AC_{[d]}^{12\text{-}3}(x,y)=\prod_{i=1}^{d}\left(1-\frac{x^iy}{\prod_{j=1}^{i-1}(1-x^jy)}\right)^{-1}.$$

Proof. Separating how the composition begins we obtain

$$\begin{aligned} AC_{[d]}^{12\text{-}3}(i|x,y) &= x^iy+\textstyle\sum_{j=1}^{i}AC_{[d]}^{12\text{-}3}(ij|x,y)+\sum_{j=i+1}^{d}AC_{[d]}^{12\text{-}3}(ij|x,y) \\ &= x^iy+x^iy\Big(\textstyle\sum_{j=1}^{i}AC_{[d]}^{12\text{-}3}(j|x,y) \\ &\qquad\qquad +\textstyle\sum_{j=i+1}^{d}x^jyAC_{[j]}^{12\text{-}3}(x,y)\Big). \end{aligned}$$

Note that in the last sum, the set of parts for the composition is restricted from $[d]$ to $[j]$ to guarantee avoidance of 12-3. From this recursion, we get that the generating function

$$G_d(i)=AC_{[d]}^{12\text{-}3}(i|x,y)-AC_{[d-1]}^{12\text{-}3}(i|x,y)$$

satisfies

$$G_d(i)=x^iy\sum_{j=1}^{i}G_d(j)+x^iy\cdot x^dyAC_{[d]}^{12\text{-}3}(x,y),$$

and solving for $G_d(i)$ leads to

$$G_d(i)=\frac{x^iy}{1-x^iy}\left(\sum_{j=1}^{i-1}G_d(j)+x^dyAC_{[d]}^{12\text{-}3}(x,y)\right).$$

It is not hard to prove by induction on i that

$$G_d(i)=\frac{x^{i+d}y^2AC_{[d]}^{12\text{-}3}(x,y)}{\prod_{j=1}^{i}(1-x^jy)},$$

for all $i=1,2,\ldots,d-1$. The induction step uses Lemma 4.1. Also, for $i=d$, we obtain from the definition that

$$\begin{aligned} G_d(d) &= AC_{[d]}^{12\text{-}3}(d|x,y)-AC_{[d-1]}^{12\text{-}3}(d|x,y) \\ &= x^dyAC_{[d]}^{12\text{-}3}(x,y)-0=x^dyAC_{[d]}^{12\text{-}3}(x,y). \end{aligned}$$

Therefore, summing over all possible values $i = 1, 2, \dots, d$ we obtain

$$AC_{[d]}^{12\text{-}3}(x,y) - AC_{[d-1]}^{12\text{-}3}(x,y) = x^d y \left(1 + \sum_{i=1}^{d-1} \frac{x^i y}{\prod_{j=1}^{i}(1 - x^j y)}\right) AC_{[d]}^{12\text{-}3}(x,y),$$

which by Lemma 4.1 is equivalent to

$$AC_{[d]}^{12\text{-}3}(x,y) - AC_{[d-1]}^{12\text{-}3}(x,y) = \frac{x^d y}{\prod_{j=1}^{d-1}(1 - x^j y)} AC_{[d]}^{12\text{-}3}(x,y).$$

Hence, for all $d \geq 1$ we have

$$AC_{[d]}^{12-3}(x,y) = \left(1 - \frac{x^d y}{\prod_{j=1}^{d-1}(1 - x^j y)}\right)^{-1} AC_{[d-1]}^{12-3}(x,y).$$

Iterating the above recurrence relation d times together with the initial condition $AC_{\emptyset}^{12-3}(x,y) = 1$ we get the desired result. □

Example 4.3 *Now we can easily obtain the generating function for the number of compositions of n that avoid the pattern* 12-3 *as*

$$AC_{\mathbb{N}}^{12-3}(x,1) = \prod_{i \geq 1} \left(1 - \frac{x^i}{\prod_{j=1}^{i-1}(1 - x^j)}\right)^{-1}.$$

The corresponding sequence for the number of compositions of n that avoid 12-3 *for $n = 0$ to $n = 20$ is given by* 1, 1, 2, 4, 8, 16, 31, 60, 114, 215, 402, 7464, 1375, 2520, 4593, 8329, 15036, 27027, 48389, 86314 *and* 153432.

Example 4.4 *(see [3, Theorem 3.6]) Theorem 4.2 for $x = 1$ and $d = k$ we get that the generating function for the number k-ary words of length n that avoid* 12-3 *is given by*

$$AC_{[k]}^{12\text{-}3}(1,y) = \prod_{i=1}^{k} \left(1 - \frac{y}{(1-y)^{i-1}}\right)^{-1} = \prod_{i=0}^{k-1} \left(1 - \frac{y}{(1-y)^{i}}\right)^{-1}.$$

Theorem 4.5 *The generating function for the number of compositions of n with m parts in $[d]$ that avoid* 23-1 *is given by*

$$AC_{[d]}^{23\text{-}1}(x,y) = \prod_{i=1}^{d} \left(1 - \frac{x^i y}{\prod_{j=i+1}^{d}(1 - x^j y)}\right)^{-1}.$$

Proof. Let $[i,j] = \{i, i+1, \dots, j\}$ and let σ be any composition of n with m parts in $[d]$ that avoids 23-1. Then σ either does not contain the part 1, or σ can be decomposed as

$$\sigma^{(1)} 1 \sigma^{(2)} 1 \cdots \sigma^{(s)} 1 \sigma',$$

where $\sigma^{(i)}$ and σ' have parts in $[2,d]$, each $\sigma^{(i)}$ avoids 12 and σ' avoids 23-1. The generating function is given by $\left(xyAC_{[2,d]}^{12}(x,y)\right)^s AC_{[2,d]}^{23\text{-}}(x,y)$ for $s \geq 1$. Altogether,

$$AC_{[d]}^{23\text{-}}(x,y) = AC_{[2,d]}^{23\text{-}}(x,y) + \frac{xyAC_{[2,d]}^{12}(x,y)}{1 - xyAC_{[2,d]}^{12}(x,y)} AC_{[2,d]}^{23\text{-}}(x,y),$$

which is equivalent to

$$AC_{[d]}^{23\text{-}1}(x,y)=\frac{AC_{[2,d]}^{23\text{-}}(x,y)}{1-xyAC_{[2,d]}^{12}(x,y)}.$$

Using the above recurrence d times we obtain that

$$AC_{[d]}^{23\text{-}1}(x,y)=\prod_{i=1}^{d}\frac{1}{1-x^iyAC_{[i+1,d]}^{12}(x,y)}.$$

Using the fact that $AC_{[i+1,d]}^{12}(x,y)=\prod_{j=i+1}^{d}(1-x^jy)^{-1}$ we complete the proof. □

Example 4.6 *Taking the limit $d\to\infty$ together with the substitution $y=1$ in Theorem 4.5 we get that the generating function for the number of compositions of n that avoid the pattern* 23-1 *is given by*

$$AC_{\mathbb{N}}^{23-1}(x,1)=\prod_{i\geq 1}\left(1-\frac{x^i}{\prod_{j\geq i+1}(1-x^j)}\right).$$

The sequence for the number of compositions of n that avoid 23-1 *for $n=0$ to $n=20$ is given by* 1, 1, 2, 4, 8, 16, 31, 61, 118, 228, 440, 846, 1623, 3111, 5955, 11385, 21752, 41530, 79250, 151161 *and* 288224.

Example 4.7 *(see [3, Theorem 3.6]) Theorem 4.2 for $x=1$ and $d=k$ we get that the generating function for the number k-ary words of length n that avoid* 23-1 *is given by*

$$AC_{[k]}^{23\text{-}1}(1,y)=\prod_{i=1}^{k}\left(1-\frac{y}{(1-y)^{k-i}}\right)^{-1}=\prod_{i=0}^{k-1}\left(1-\frac{y}{(1-y)^{i}}\right)^{-1}.$$

Using arguments similar to those in the proofs of Theorems 4.2 and 4.5 we obtain a recursive result. Finding an explicit expression for $AC_{[d]}^{13-2}(x,y)$ remains an open question.

Lemma 4.8 *The generating function $AC_{[d]}^{13-2}(x,y)$ for the number of compositions of n with m parts in $[d]$ that avoid* 13-2 *satisfies*

$$AC_{[d]}^{13-2}(x,y)=1+\sum_{i=1}^{d}AC_{[d]}^{13-2}(i|x,y),$$

where

$$AC_{[d]}^{13-2}(i|x,y)=\ x^iy\Big(1+\textstyle\sum_{j=1}^{i+1}AC_{[d]}^{13-2}(j|x,y)+\sum_{j=i+2}^{d}AC_{\{1,\dots,i,j,\dots,d\}}^{13-2}(j|x,y)\Big).$$

Example 4.9 *Using $d=15$ and $y=1$ in Lemma 4.8 we get that the sequence for the number of compositions of n that avoid* 13-2 *for $n=0$ to $n=15$ is given by* 1, 1, 2, 4, 8, 16, 31, 60, 115, 218, 411, 770, 1434, 2656, 4897 *and* 8991.

5 Conclusion

We have completely classified avoidance of the permutation patterns of type $(2, 1)$ and given explicit generating functions for all but the pattern 13-2, which remains an open question. A natural extension is to consider avoidance of multi-permutation patterns of this type, namely the patterns 11-1, 11-2, 12-1, 12-2, 21-1, 21-2 and 22-1. We already have some partial results for these patterns, such as the classification according to Wilf-equivalence, and plan to describe the complete results in our forthcoming book.

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