Enumerating finite set partitions according to the number of connectors

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Abstract

Let $P(n,k)$ denote the set of partitions of $[n]=\{1,2,\ldots,n\}$ containing exactly k blocks. Given a partition $\Pi=B_1/B_2/\cdots/B_k\in P(n,k)$ in which the blocks are listed in increasing order of their least elements, let $\pi=\pi_1\pi_2\cdots\pi_n$ denote the *canonical sequential form* wherein $j\in B_{\pi_j}$ for all $j\in[n]$. In this paper, we supply an explicit formula for the generating function which counts the elements of $P(n,k)$ according to the number of strings $k1$ and $r(r+1)$, taken jointly, occurring in the corresponding canonical sequential forms. A comparable formula for the statistics on $P(n,k)$ recording the number of strings $1k$ and $r(r-1)$ is also given which may be extended to strings $r(r-1)\cdots(r-m)$ of arbitrary length using linear algebra. In addition, we supply algebraic and combinatorial proofs of explicit formulas for the total number of occurrences of $k1$ and $r(r+1)$ within all the members of $P(n,k)$.

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1 Introduction

A partition of $[n]=\{1,2,\ldots,n\}$ is a decomposition of $[n]$ into nonempty subsets called *blocks*. A partition with k blocks is also called a k-partition and is denoted by $B_1/B_2/\cdots/B_k$, where blocks are listed in increasing order of their least elements. The set of all partitions of $[n]$ with exactly k blocks will be denoted by $P(n,k)$ and has cardinality given by $S(n,k)$, the well-known Stirling number of the second kind [13].

Let us recall two statistics on $P(n,k)$ which were introduced in [8].

Definition 1. Let $\Pi=B_1/B_2/\cdots/B_k$ denote a member of $P(n,k)$, where $k>1$.

(*i*) A pair $(a,a+1),\ a\in[n],$ is called a (linear) connector if $a\in B_i$ and $a+1\in B_{i+1},\ i\in[k-1]$.

(*ii*) A pair $(a,a+1),\ a\in[n],$ is called a *circular* connector if $a\in B_i$ and $a+1\in B_{i+1},\ i\in[k-1]$, or $a\in B_k,\ a+1\in B_1$; the pair $(n,1)$ is a circular connector provided $n\in B_k$.

(*iii*) We define $con(\pi)$ as the number of connectors in π, and $ccon(\pi)$ as the number of cicular connectors in π.

Furthermore, we denote the difference $ccon(\pi) - con(\pi)$ by $cir(\pi)$; note that $cir(\pi)$ counts the number of elements $a \in [n]$ belonging to B_k for which $a+1 \in B_1$ (with n being counted whenever $n \in B_k$). For example, if $\Pi = 1,6,8/2,4/3/5,7,9 \in P(9,4)$, then $con(\pi) = 2$, which accounts for (1,2) and (2,3), $ccon(\pi) = 5$, with (5,6), (7,8), and (9,1) the corresponding circular connectors which aren't linear, and $cir(\pi) = 3$. Note that circular connectors are connectors when the blocks of a partition are arranged on a circle. The concept of circular connectors may be viewed as an extension to set partitions of the concept of *circular combinations*, the study of which was pioneered by Kaplansky [5]: subsets of $[n]$ are counted according to the number of pairs of elements a, b satisfying $b - a \equiv 1 \pmod{n}$. See also the recent related papers by Chen, Wang, and Zhang [1] and Guo [3]. The tables below compare the arrays for $0 \leqslant \ell \leqslant n \leqslant 5$ obtained by counting the combinations of $[n]$ according to the number of pairs of consecutive elements (taken in a line or on a circle) and by counting the number of partitions of $[n]$ according to the number of linear and circular connectors.

$n\backslash\ell$	0	1	2	3	4
1	2				
2	3	1			
3	5	2	1		
4	8	5	2	1	
5	13	9	6	2	1

$n\backslash\ell$	0	1	2	3	4	5
1	2					
2	3	0	1			
3	4	3	0	1		
4	7	4	4	0	1	
5	11	10	5	5	0	1

Table 1.1: Number of linear and circular combinations of $[n]$ with ℓ pairs of consecutive elements, where $n = 1, 2, 3, 4, 5$.

$n\backslash\ell$	0	1	2	3	4
1	1				
2	1	1			
3	1	3	1		
4	1	7	6	1	
5	1	16	25	9	1

$n\backslash\ell$	0	1	2	3	4	5
1	1					
2	1	0	1			
3	1	0	3	1		
4	1	0	8	4	2	
5	1	1	20	15	14	1

Table 1.2: Number partitions of $[n]$ with ℓ linear connectors and ℓ circular connectors, where $n = 1, 2, 3, 4, 5$.

The study of partition statistics is often motivated by the analogous equidistribution question arising in the study of permutations statistics (see, e.g., [15]). Accompanying this direction is the invention of q-analogues for various enumerative functions including the Stirling numbers. Influential papers in this direction include papers by Sagan [10] and Wachs and White [14]. Here we consider some particular cases of the general problem of counting the members of a partition class having a restriction imposed on the relative positions of elements within and among blocks (see, e.g., [2]).

In what follows, we will often represent a partition $B_1/B_2/\cdots/B_k$ of $[n]$ in the *canonical sequential form* $\pi = \pi_1\pi_2\cdots\pi_n$ such that $j \in B_{\pi_j}$ for all $j \in [n]$ (see, e.g., [13] for details). For example, if $\Pi = 1,4/2,5,7/3/6,8$ is a partition of [8], then its canonical sequential form is $\pi = 12312424$ and in such case we write $\Pi = \pi$. Note that $\pi = \pi_1\pi_2\cdots\pi_n \in P_{n,k}$ is a *restricted growth function* from $[n]$ to $[k]$ (see, e.g., [9] for details), meaning that it satisfies the following three properties: (i) $\pi_1 = 1$, (ii) π is onto $[k]$, and (iii) $\pi_{i+1} \leqslant \max\{\pi_1, \pi_2, \ldots, \pi_i\} + 1$ for all i, $1 \leqslant i \leqslant n-1$.

A *rise* in a partition $\Pi = B_1/B_2/\cdots/B_k$ of $[n]$ is defined as a rise in the corresponding sequential form $\pi = \pi_1\pi_2\cdots\pi_n$, i.e., an index $i < n$ such that $\pi_i < \pi_{i+1}$ (see, e.g., [7]). Note that a connector $(a, a+1)$ in Π corresponds to a rise at a in π of size 1. If Π is the partition given in the previous paragraph, then there are rises at positions 1, 2, 4, 5, and 7 in π, but only the first three rises correspond to connectors in Π. Thus, if $\pi = \pi_1\pi_2\cdots\pi_n \in P(n,k)$, then $con(\pi)$ gives the total number of rises in π of size 1 (i.e., the number of strings $r(r+1)$ in π with $r < k-1$), $cir(\pi)$ gives the number of occurrences of the string $k1$ (including $\pi_n = k$, $\pi_1 = 1$), and $ccon(\pi)$ gives the number of occurrences of strings of either type.

Counting the number of words containing a set of given strings as substrings a certain number of times is a classical problem in enumerative combinatorics. The problem can, for example, be attacked using the transfer matrix method (see [12, Section 4.7]). In particular, it is a well-known fact that the generating function for words avoiding a fixed number of substrings is always rational. For example, the generating function for the number of words in $[3]^n$ in which neither 22 nor 13 appear as two consecutive digits is given by $\frac{3+x-x^2}{1-2x-x^2+x^3}$.

Here, we consider the problem of counting the substrings $12, 23, \ldots, (k-1)k, k1$ within members of $P(n,k)$, represented *canonically* as words. Since the number of substrings increases as n and k increase and since there is a further restriction on the words (see the canonical form described above for set partitions), it does not seem possible to apply the transfer matrix method in these cases. Instead, to derive our results, we make use of certain decompositions of set partitions and introduce auxiliary functions which count certain subsets of the partitions in question. The generating functions that we find in the end by this method, nonetheless, are rational in all cases.

In particular, we first find an explicit formula for the generating function counting k-partitions of $[n]$, where k is fixed, according to the *cir* and *con* statistics, taken jointly, as requested in [8], which generalizes several of the formulas found there. In addition, taking special values in this formula yields expressions for generating functions which count certain restricted classes of partitions (see, e.g., Corollary 2.8 below). We also find a comparable formula for the generating function which counts members of $P(n,k)$ according to the number of occurrences of the strings $1k$ and $r(r-1)$, with $r > 1$, which further extends the results in [8]. Using linear algebra, this formula may be generalized to count members of $P(n,k)$ according to the number of occurrences of the string $r(r-1)\cdots(r-m)$ of arbitrary length. We note here that there does not appear to be an analogous formula for strings of the form $r(r+1)\cdots(r+m)$. Furthermore, we provide algebraic and combinatorial proofs of explicit expressions for the total value of the *cir* and *con* statistics taken over all the members of $P(n,k)$ and provide a bijective proof for a related recurrence, which was requested in [8]. For other examples of the general problem of enumerating set partitions with respect to special patterns, see, e.g., Sagan [11], Klazar [6], and Jelinik and Mansour [4] as well as the references contained within [4].

2 Counting rises

2.1 A joint generating function

In this section, we derive an explicit formula for the generating function of the joint distribution polynomial for the *cir* and *con* statistics on $P(n,k)$, as requested in [8], from which several prior results as well as some new will follow directly as special cases.

If $\alpha \in [k]^n$, then let $cir(\alpha)$ and $con(\alpha)$ denote, respectively, the number of strings in α of the forms $k1$ and $r(r+1)$ with $r < k$. Given $n \geqslant 0$ and $k \geqslant 2$, let $F_{n,k}(p,q)$ be the joint distribution polynomial on words $\alpha \in [k]^n$ for the statistic values $cir(k\alpha 1)$ and $con(k\alpha 1)$, i.e.,

$$F_{n,k}(p,q) := \sum_{\alpha \in [k]^n} p^{cir(k\alpha 1)} q^{con(k\alpha 1)},$$

and let $F_k(x,p,q)$ be the corresponding generating function, i.e.,

$$F_k(x,p,q) := \sum_{n \geqslant 0} F_{n,k}(p,q) x^n.$$

Lemma 2.1. *If $k \geqslant 2$, then the generating function $F_k(x,p,q)$ for the number of k-ary words α of length n according to the number of occurrences of the strings $k1$ and $r(r+1)$, with $r < k$, in $k\alpha 1$ is given by*

$$F_k(x,p,q) = \frac{p + x[(k-1) - (k-2+q)p] + x(1-p)\left(\frac{1-x^{k-1}(q-1)^{k-1}}{1+x(p-q)-x^k(q-1)^{k-1}(p-1)}\right)R}{1 - x(k-1+q) - x\left(\frac{1-x^k(q-1)^k}{1+x(p-q)-x^k(q-1)^{k-1}(p-1)}\right)R}, \tag{2.1}$$

where $R := (p-1) + x(1-q) + x(1-p)(k-2+q)$.

Proof. We first introduce some auxiliary generating functions. If $a \in [k]$, then let $F(a) = F_k(x,p,q|a)$ denote the generating function for the k-ary words $\alpha = \alpha_1\alpha_2\cdots\alpha_n$ with $\alpha_1 = a$ according to the number of occurrences of the strings $k1$ and $r(r+1)$ in the word $k\alpha 1$. From the definitions, we have

$$F = p + \sum_{a=1}^{k} F(a), \tag{2.2}$$

where $F = F_k(x,p,q)$. If $a \in [k]$, then let $G_a = G_a(x,p,q)$ denote the joint generating function for k-ary words α according to the number of occurrences of the strings $k1$ and $r(r+1)$ in $a\alpha 1$. From the definitions, we have

$$F(1) = xpG_1, \quad F(a) = xG_a, \quad 2 \leqslant a \leqslant k-1, \quad \text{and} \quad F(k) = xF = xG_k. \tag{2.3}$$

We also have from the definitions,

$$G_a = 1 + \sum_{i=1, i \neq a+1}^{k} xG_i + xqG_{a+1}, \qquad 1 \leqslant a \leqslant k-1, \tag{2.4}$$

with $G_k = F$.

Combining (2.2) and (2.3) yields

$$F - p = x\sum_{i=1}^{k} G_i + x(p-1)G_1. \tag{2.5}$$

Combining (2.4) and (2.5) yields

$$G_a = 1 + (F - p - x(p-1)G_1 - xG_{a+1}) + xqG_{a+1}, \tag{2.6}$$

which we may rewrite as

$$G_a - x(q-1)G_{a+1} = F + 1 - p - x(p-1)G_1, \qquad 1 \leqslant a \leqslant k-1, \tag{2.7}$$

with $G_k = F$. Multiplying the a^{th} equation of (2.7) by $[x(q-1)]^{a-1}$ and adding equations yields

$$G_1 - (x(q-1))^{k-1}F = (F + 1 - p - x(p-1)G_1)\left[\frac{1-(x(q-1))^{k-1}}{1-x(q-1)}\right]. \tag{2.8}$$

After some algebra, equation (2.8) may be rewritten as

$$G_1 = \left(\frac{1 - x^k(q-1)^k}{1 + x(p-q) - x^k(q-1)^{k-1}(p-1)}\right)F + \frac{(1-p)(1-x^{k-1}(q-1)^{k-1})}{1 + x(p-q) - x^k(q-1)^{k-1}(p-1)}. \tag{2.9}$$

In the lemma that follows we show that

$$F = p + x[(k-1) - (k-2+q)p] + x(k-1+q)F + xRG_1, \tag{2.10}$$

where $R := (p-1) + x(1-q) + x(1-p)(k-2+q)$. Solving (2.9) and (2.10) simultaneously for G_1 and F yields Lemma 2.1. □

Lemma 2.2. *We have*

$$F = p + x[(k-1) - (k-2+q)p) + x(k-1+q)F + xRG_1, \tag{2.11}$$

where F, R, and G_1 are as defined above.

Proof. Adding the equations in (2.4) yields

$$\sum_{i=1}^{k} G_i = (k-1) + F + x(k-1)G_1 + [x(k-2) + xq]\sum_{i=2}^{k} G_i. \tag{2.12}$$

Rewriting the sum of the last two terms on the right side of (2.12), we have

$$\begin{aligned} x(k-1)G_1 + [x(k-2) + xq]\sum_{i=2}^{k} G_i &= x(1-q)G_1 + x(k-2+q)\sum_{i=1}^{k} G_i \\ &= x(1-q)G_1 + (k-2+q)[F - p - x(p-1)G_1], \end{aligned}$$

by (2.5), which implies

$$\sum_{i=1}^{k} G_i = (k-1) + F + x(1-q)G_1 + (k-2+q)[F - p - x(p-1)G_1]. \tag{2.13}$$

On the other hand, we have

$$\sum_{i=1}^{k} G_i = \frac{F-p}{x} - (p-1)G_1, \tag{2.14}$$

by (2.5). Equating the expressions in (2.13) and (2.14) yields (2.11). □

We can now compute the joint generating function $H_k(x,p,q)$ for k-partitions of $[n]$ according to the values of the *cir* and *con* statistics, i.e.,

$$H_k(x,p,q) := \sum_{n\geqslant 0} x^n \sum_{\pi\in P(n,k)} p^{cir(\pi)} q^{con(\pi)}.$$

Let $BW_j(x,q)$ denote the generating function for the number of j-ary words α of length n according to the value of $con(j\alpha(j+1))$. From the proof of Theorem 2.1 in [8], we have

$$BW_j(x,q) = q-1+\frac{1-x^{j+1}(q-1)^{j+1}}{1-x(j+q)+x\frac{1-x^{j+1}(q-1)^{j+1}}{1-x(q-1)}}, \qquad j\geqslant 1. \tag{2.15}$$

Since each partition π with exactly k blocks can decomposed uniquely as

$$\pi = 1w^{(1)}2w^{(2)}\cdots kw^{(k)},$$

where $w^{(i)}$ is a word over the alphabet $[i]$, combining (2.1) and (2.15) yields an explicit formula for $H_k(x,p,q)$, as requested in [8].

Theorem 2.3. *If $k\geqslant 2$, then the generating function $H_k(x,p,q)$ for the number of partitions of $[n]$ with k blocks according to the value of the cir and con statistics is given by*

$$H_k(x,p,q) = x^k F_k(x,p,q)\prod_{j=1}^{k-1} BW_j(x,q), \tag{2.16}$$

where $F_k(x,p,q)$ and $BW_j(x,q)$ are given by (2.1) *and* (2.15)*, respectively.*

Several of the formulas shown in [8] now follow as special cases of (2.16). For example, taking $p=1$ in (2.16) yields the generating function $C_k(x,q)$ for the number of partitions of $[n]$ with k blocks according to the value of *con*.

Corollary 2.4. *If $k\geqslant 2$, then the generating function $C_k(x,q)$ for the number of partitions of $[n]$ with k blocks according to the number of occurrences of the string $r(r+1)$, with $r\leqslant k-1$, is given by*

$$C_k(x,q) = \frac{x^k}{1-x\sum_{i=1}^k \frac{1-x^i(q-1)^i}{1-x(q-1)}}\prod_{j=1}^{k-1}\left(q-1+\frac{1-x^{j+1}(q-1)^{j+1}}{1-x(j+q)+x\frac{1-x^{j+1}(q-1)^{j+1}}{1-x(q-1)}}\right). \tag{2.17}$$

Taking $q=1$ in (2.16) yields the generating function $AC_k(x,q)$ for the number of partitions of $[n]$ with k blocks according to the value of *cir*.

Corollary 2.5. *If $k\geqslant 2$, then the generating function $AC_k(x,q)$ for the number of partitions of $[n]$ with k blocks according to the value of the cir statistic is given by*

$$AC_k(x,q) = \frac{x^k}{(1-x)(1-2x)\cdots(1-(k-1)x)}\left(\frac{q+(k-2)(1-q)x}{1-kx+(1-q)x^2}\right). \tag{2.18}$$

Corollaries 2.4 and 2.5 occur, respectively, as Theorems 2.1 and 5.6 in [8].

Taking $p=q$ in Theorem 2.3 yields an explicit formula for the generating function $CC_k(x,q)$ for the number of partitions of $[n]$ with k blocks according to the value of *ccon*.

Corollary 2.6. *If $k \geqslant 2$, then the generating function $CC_k(x,q)$ for the number of partitions of $[n]$ with k blocks according to the value of the ccon statistic is given by*

$$CC_k(x,q) = x^k BV_k(x,q) \prod_{j=1}^{k-1} BW_j(x,q), \tag{2.19}$$

where $BW_j(x,q)$ is given by (2.15) and

$$BV_k(x,q) := \frac{q + x[(k-1)-(k-2+q)q] + x(1-q)^2[-1+x(k-1+q)]\left(\frac{1-x^{k-1}(q-1)^{k-1}}{1-x^k(q-1)^k}\right)}{1-x(k-2+2q)-x^2(1-q)(k-1+q)}.$$

We note that formula (2.19) slightly corrects the expression for $CC_k(x,q)$ given in Theorem 2.2 of [8].

Let $EA_k(x)$ be the generating function for the number of k-partitions of $[n]$ containing exactly one string of the form $a(a+1)$ (necessarily 12) and at least one string of the form $k1$ (which may be $\pi_n = k$, $\pi_1 = 1$). Such partitions are termed *essentially arc-connected* in [8]. Then $EA_k(x)$ can be obtained from $H_k(x,p,q)$ via the relation

$$EA_k(x) = \frac{H_k(x,p,q)}{q}\ |_{p=1,q=0} - \frac{H_k(x,p,q)}{q}\ |_{p=q=0}\ . \tag{2.20}$$

Simplifying the right-hand side of (2.20) yields the following explicit formula for $EA_k(x)$ which occurs as Theorem 5.9 in [8].

Corollary 2.7. *If $k \geqslant 2$, then the generating function $EA_k(x)$ for the number of essentially arc-connected k-partitions of $[n]$ is given by*

$$EA_k(x) = (-1)^k x^{2k-2} \frac{(1-(k-3)x-(k-1)x^2+(-x)^{k+1})^2}{(1-x^2)(1-(k-1)x)(1-(-x)^k)} \frac{\prod_{j=2}^{k-1}(1-j-jx-(-x)^j)}{\prod_{j=2}^{k}(1-(j-2)x-jx^2+(-x)^{j+2})}. \tag{2.21}$$

One may count other classes of partitions using (2.16) above. For example, let us call a k-partition of $[n]$ *non-connected* if it has *cir* value 0 and *con* value 1 (i.e., if the only connector is the trivial connector 12 corresponding to the left-most occurrence of 2). If $N_k(x)$ is the generating function for the number of non-connected k-partitions of $[n]$, then $N_k(x) = \frac{H_k(x,p,q)}{q}\ |_{p=q=0}$, which implies the following result, using (2.16).

Corollary 2.8. *If $k \geqslant 2$, then the generating function $N_k(x)$ for the number of non-connected k-partitions of $[n]$ is given by*

$$N_k(x) = \frac{x^k}{1-x}\left(\frac{(k-2)x+(k-1)x^2-(-x)^k}{(1+x)(1-(k-1)x)(1-(-x)^k)}\right)\prod_{j=2}^{k-1}\frac{(j-1)x+jx^2-(-x)^{j+1}}{1-(j-2)x-jx^2+(-x)^{j+2}}. \tag{2.22}$$

Let us call a k-partition of $[n]$ *truly line-connected* if it has *cir* value 0 and if it contains at least one occurrence of $r(r+1)$ for some $r \in [k-1]$ in addition to the 12 corresponding to the left-most occurrence of 2 (i.e., has *con* value of at least two). If $T_k(x)$ denotes the corresponding generating function, then $T_k(x)$ may be given explicitly by $T_k(x) = AC_k(x,0) - N_k(x)$, where $AC_k(x,q)$ is given by (2.18) above.

2.2 Some combinatorial results

In this section, we find explicit formulas for the total value of the *con*, *cir*, and *ccon* statistics taken over all the members of $P(n,k)$, providing both algebraic and combinatorial proofs. We first consider *con*. Taking the derivative of $C_k(x,q)$ in (2.17) above with respect to q and setting $q=1$ gives

$$\frac{d}{dq}C_k(x,q)\mid_{q=1} = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}\left(\frac{(k-1)x^2}{1-kx}+\sum_{j=1}^{k-1}(1-jx)(1+\frac{x-x^2}{(1-jx)^2})\right)$$
$$= C_k(x,1)\left(\frac{(k-1)x^2}{1-kx}+(k-1)x+\sum_{j=1}^{k-1}\left[(1-jx)+\frac{(j-1)x^2}{1-jx}\right]\right).$$

Extracting coefficients of x^n and using the fact (see, e.g., p. 46 of [12])

$$C_k(x,1)=\sum_{n\geqslant k}S(n,k)x^n=\frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}$$

implies

$$[x^n]\frac{d}{dq}C_k(x,q)\mid_{q=1} = (k-1)S(n-1,k)+(k-1)\sum_{i=0}^{n-2-k}k^iS(n-2-i,k)$$
$$+\sum_{j=1}^{k-1}(S(n,k)-jS(n-1,k))+\sum_{j=1}^{k-1}(j-1)\sum_{i=0}^{n-2-k}j^iS(n-2-i,k)$$
$$=(k-1)(S(n-1,k)+S(n,k))-\binom{k}{2}S(n-1,k)$$
$$+\sum_{j=2}^{k}(j-1)\sum_{i=0}^{n-2-k}j^iS(n-2-i,k).$$

This yields the following result.

Theorem 2.9. *If $n\geqslant k>1$, then the total con value of all the members of $P(n,k)$ is given by*

$$(k-1)S(n,k)-\binom{k-1}{2}S(n-1,k)+\sum_{j=2}^{k}(j-1)\sum_{i=0}^{n-2-k}j^iS(n-2-i,k).$$

Proof. We may also establish this result directly by counting the total number of strings of the form $r(r+1)$ occurring within all the members of $P(n,k)$ as follows. We will call a letter t *initial* if it represents the first occurrence of its type in a left-to-right scan of $\pi\in P(n,k)$, represented canonically. Note first that there are $(k-1)S(n-1,k)$ strings $j(j+1)$ within the members of $P(n,k)$ in which the letter $j+1$ is initial but j is not, as seen upon inserting the letter j directly before the first occurrence of $j+1$ within a member of $P(n-1,k)$ for any $j\in[k-1]$.

We now argue that the total number of strings $r(r+1)$ in which both the letters r and $r+1$ are initial is given by $(k-1)S(n,k)-\binom{k}{2}S(n-1,k)$. First note that there are clearly a total of $(k-1)S(n,k)$

initial letters less than k occurring within all the members of $P(n,k)$. From this, we subtract the number of strings of the form ri, within all members of $P(n,k)$, where $r \in [k-1]$ is initial and $i \in [r]$. For each fixed $r \in [k-1]$, there are $rS(n-1,k)$ such strings within all the members of $P(n,k)$ (upon inserting a letter $i \in [r]$ directly after the first occurrence of r within a member of $P(n-1,k)$) and thus $\sum_{r=1}^{k-1} rS(n-1,k) = \binom{k}{2}S(n-1,k)$ strings in all.

To complete the proof, we must show that the total number of strings $r(r+1)$ in which neither r nor $r+1$ is initial is given by

$$\sum_{j=2}^{k}(j-1)\sum_{i=0}^{n-2-k} j^i S(n-2-i,k).$$

Given i and j, where $2 \leqslant i \leqslant n-2-k$ and $2 \leqslant j \leqslant k$, consider all the members of $P_{n,k}$ which may be decomposed uniquely as

$$\pi = \pi' j\alpha\beta, \tag{2.23}$$

where π' is a partition with $j-1$ blocks, α is a word of length $i+2$ in the alphabet $[j]$ whose last two letters form a string $r(r+1)$, and β is possibly empty. For example, if $i=2$, $j=4$, and $\pi = 121324132345 \in P_{12,5}$, then $\pi' = 12132$, $\alpha = 1323$, and $\beta = 45$. The total number of strings $r(r+1)$ in which neither letter is initial can then be obtained by finding the number of partitions which may be expressed as in (2.23) for each i and j and then summing over all possible values of i and j. Note that there are $(j-1)j^iS(n-2-i,k)$ members of $P_{n,k}$ which may be expressed as in (2.23) since there are j^i choices for the first i letters of α, $j-1$ choices for the final two letters in α (as the last letter must exceed its predecessor by one), and $S(n-2-i,k)$ choices for the remaining letters $\pi' j\beta$ which necessarily constitute a partition of an $(n-2-i)$-set into k blocks. □

Next, we consider the *cir* statistic. Taking the q-partial derivative of both sides of (2.18) and setting $q=1$ gives

$$\begin{aligned}\frac{d}{dq}AC_k(x,q)\mid_{q=1} &= \frac{x^k}{(1-x)(1-2x)\cdots(1-(k-1)x)}\left[\frac{1-(k-2)x}{1-kx}+\frac{x^2}{(1-kx)^2}\right]\\ &= AC_k(x,1)\left[1-(k-2)x+\frac{x^2}{1-kx}\right],\end{aligned}$$

which implies

$$\begin{aligned}[x^n]\frac{d}{dq}AC_k(x,q)\mid_{q=1} &= S(n,k)-(k-2)S(n-1,k)+\sum_{i=0}^{n-2-k}k^iS(n-2-i,k)\\ &= S(n-1,k-1)+2S(n-1,k)+\sum_{i=0}^{n-2-k}k^iS(n-2-i,k),\end{aligned}$$

where we have used the recurrence $S(n,k) = S(n-1,k-1) + kS(n-1,k)$. This yields the following result.

Theorem 2.10. *If $n \geqslant k > 1$, then the total cir value of all the members of $P(n,k)$ is given by*

$$S(n-1,k-1)+2S(n-1,k)+\sum_{i=0}^{n-2-k}k^iS(n-2-i,k).$$

Proof. We can show this directly by counting the total number of occurrences of the string $k1$ within all the members of $P(n,k)$ (counting $\pi_n = k$, $\pi_1 = 1$). We first count all $k1$ strings in which the k corresponds to the last letter of a partition. In this case, there are $S(n-1,k-1)$ such strings in which the k is initial and $S(n-1,k)$ strings in which it is not. There are also $S(n-1,k)$ strings $k1$ in which the k is initial but does not correspond to the final letter of a partition (simply insert 1 directly after the first occurrence of k within a member of $P(n-1,k)$).

To complete the proof, we must show that the total number of strings $k1$ in which neither k nor 1 is initial is given by $\sum_{i=0}^{n-2-k} k^i S(n-2-i,k)$. Given i, where $0 \leqslant i \leqslant n-2-k$, consider all the members of $P(n,k)$ which may be decomposed uniquely as

$$\pi = \pi' k \alpha k 1 \beta, \tag{2.24}$$

where π' is a partition with $k-1$ blocks, α is a k-ary word of length i, and β is possibly empty. The total number of strings $k1$ in which neither letter is initial can then be obtained by finding the number of partitions which may be expressed as in (2.24) for each i and then summing over all i. And there are $k^i S(n-2-i,k)$ members of $P(n,k)$ which may be expressed as in (2.24) as there are k^i choices for α and $S(n-2-i,k)$ choices for the letters $\pi' k \beta$. □

Taking the derivative in (2.19) or adding the expressions in Theorems 2.9 and 2.10 yields the following corollary.

Corollary 2.11. *If $n \geqslant k > 1$, then the total ccon value of all the members of $P(n,k)$ is given by*

$$kS(n,k) - \left(\binom{k}{2} - 1\right) S(n-1,k) + \sum_{j=2}^{k} (j-1+\delta_{j,k}) \sum_{i=0}^{n-2-k} j^i S(n-2-i,k).$$

We conclude this section by providing a direct proof of a related recurrence. The partition $\pi = \pi_1\pi_2\cdots\pi_n \in P(n,k)$, expressed canonically, is said to be *purely line-connected* if it contains no occurrences of the string $k1$ (counting $\pi_n = k$, $\pi_1 = 1$), i.e., if its *cir* value is zero. Let $\ell c(n,k)$ denote the number of purely line-connected k-partitions of $[n]$. The following recurrence for $\ell c(n,k)$ was established algebraically in [8] and the question of finding a direct combinatorial proof was raised. Here we supply the requested proof.

Theorem 2.12. *Let n and k be positive integers with $k > 1$. Then*

$$\ell c(n,k) = k\ell c(n-1,k) - \ell c(n-2,k) + (k-2)S(n-2,k-1), \quad n \geqslant k+2, \tag{2.25}$$

with $\ell c(k,k) = 0$ and $\ell c(k+1,k) = k-2$.

Proof. The boundary conditions are easily seen. To prove the recurrence, let $P^*(n,k) \subseteq P(n,k)$ denote the subset of purely line-connected members, where $k \geqslant 2$ and $n \geqslant k+2$ and let $\pi = \pi_1\pi_2\cdots\pi_n \in P^*(n,k)$, represented canonically. If $\pi_{n-1} = k$, where k is initial, then $\pi_n \in [k] - \{1,k\}$, which implies that there are $(k-2)S(n-2,k-1)$ such members of $P^*(n,k)$. So assume that the first k of π occurs to the left of the $(n-1)^{st}$ position. In this case, a member $\pi \in P^*(n,k)$ may be formed from a member $\pi' \in P^*(n-1,k)$ by adding any letter $c \in [k-1]$ to the end of π' (to obtain $\pi \in P^*(n,k)$ whose second-to-last letter is less than k) or by inserting the letter $c = k$ just before the final letter of π' (to obtain $\pi \in P^*(n,k)$ whose second-to-last letter is k). The latter can be performed provided the final letter of π' is greater than 1. Note that in order for a partition $\pi' = \pi''1$ to belong to $P^*(n-1,k)$, we must have $\pi'' \in P^*(n-2,k)$. Subtracting this disallowed case, we see that there are $k\ell c(n-1,k) - \ell c(n-2,k)$ members of $P^*(n,k)$ in which the first k occurs to the left of the $(n-1)^{st}$ position, which completes the proof. □

Using (2.25), it is possible to obtain Corollary 2.5 above directly in the case when $q=0$. On the other hand, there does not appear to be a comparable recurrence for the number of essentially arc-connected k-partitions of $[n]$.

3 Counting descents

We first find an explicit formula for the generating function $D_k(x,q)$ for the number of set partitions of $[n]$ with exactly k blocks according to the number of occurrences of the string $r(r-1)$ with $r>1$. To obtain $D_k(x,q)$, we must first ascertain the generating function $WD_k(x,q)$ for the number of k-ary words of length n according to the number of occurrences of the string $r(r-1)$ with $r>1$. In order to do this, we let $WD_k(x,q|a)$ be the generating function for the number of k-ary words $\alpha_1\alpha_2\cdots\alpha_n$ of length n according to the number of strings $r(r-1)$ with $r>1$ and $\alpha_1 = a$. Clearly, from the definitions, we have

$$\begin{aligned} WD_k(x,q) &= 1 + \sum_{i=1}^{k} WD_k(x,q|i), \\ WD_k(x,q|i) &= x(WD_k(x,q) - WD_k(x,q|i-1)) + xqWD_k(x,q|i-1), \quad 2 \leqslant i \leqslant k, \end{aligned}$$

with $WD_k(x,q|1) = xWD_k(x,q)$. By induction on i, this relation implies

$$WD_k(x,q|i) = xWD_k(x,q)\frac{1-(x(q-1))^i}{1-x(q-1)}, \quad i=1,2,\ldots,k. \tag{3.1}$$

Hence, the generating function $WD_k(x,q)$ is given by

$$WD_k(x,q) = \frac{1}{1 - x\sum_{i=1}^{k}\frac{1-(x(q-1))^i}{1-x(q-1)}}. \tag{3.2}$$

Since each set partition π of $[n]$ with exactly k blocks can be decomposed as $\pi = 1w^{(1)}2w^{(2)}\cdots kw^{(k)}$ where each $w^{(i)}$ is i-ary, we see that the generating function $D_k(x,q)$ is given by

$$D_k(x,q) = \prod_{i=1}^{k} WD_i(x,q|i),$$

which, by (3.1) and (3.2), yields the following result.

Theorem 3.1. *The generating function $D_k(x,q)$ for the number of partitions of $[n]$ with k blocks according to the number of occurrences of the string $r(r-1)$ with $r>1$ is given by*

$$D_k(x,q) = \frac{x^k}{\prod_{i=1}^{k}\left(1 - x\sum_{j=1}^{i}\frac{1-(x(q-1))^j}{1-x(q-1)}\right)}\prod_{i=1}^{k}\frac{1-(x(q-1))^i}{1-x(q-1)}. \tag{3.3}$$

We now provide two generalizations of Theorem 3.1, one in terms of a bivariate generating function involving a second parameter and another allowing for strings of arbitrary length.

3.1 According to the number of strings $r(r-1)$ and $1k$

In this subsection, we extend (3.3) by considering the number of strings $r(r-1)$ together with the number of strings $1k$ on k-partitions of $[n]$ in analogy to section 2.1 above. Let $J_k(x,p,q)$ denote the generating function for the number of set partitions of $[n]$ with exactly k blocks according to the number of occurrences of the strings $1k$ and $r(r-1)$ with $r>1$. Here, we provide an explicit formula for $J_k(x,p,q)$. We only state the necessary lemmas, the proofs being similar to that given for Lemma 2.1 above. Several enumerative results will follow from our general formula as special cases. For example, if one now defines a (circular) connector of a partition $B_1/B_2/\cdots/B_k \in P(n,k)$ to be some pair $(i,i+1)$, $i<n$, for which $i\in B_j$ and $i+1\in B_{j-1}$ (where $B_0=B_k$), then taking $p=q$ in $J_k(x,p,q)$ yields an explicit formula for the generating function which enumerates members of $P(n,k)$ according to the number of connectors as defined.

Lemma 3.2. *If $k\geqslant 3$, then the generating function $WE_k(x,p,q)$ for the number of words $k\alpha$ of length n, where α is k-ary, according to the number of occurrences of the strings $1k$ and $r(r-1)$, with $r>1$, is given by*

$$WE_k(x,p,q)=\frac{[x+x^2(1-q)]\sum_{i=0}^{k-1}x^i(q-1)^i}{[1-x(k+q-1)][1-x^k(q-1)^{k-1}(p-1)]+x^2(q-p)\sum_{i=0}^{k-1}x^i(q-1)^i}. \tag{3.4}$$

Lemma 3.3. *If $k\geqslant 2$, then the generating function $WD_k^{(1)}(x,q|k)$ for the number of words $k\alpha$ of length n, where α is k-ary, according to the number of occurrences of the string $r(r-1)$, $r>1$, in $k\alpha 1$ is given by*

$$WD_k^{(1)}(x,q|k)=\frac{x^{k-1}(q-1)^{k-1}[1-x(k+q-1)]+x\sum_{i=0}^{k-1}x^i(q-1)^i}{1-x(k+q-1)+x^2(q-1)\sum_{i=0}^{k-1}x^i(q-1)^i}. \tag{3.5}$$

If $k\geqslant 3$, then the generating function $WL_k(x,p,q)$ for the number of words $(k-1)\alpha$ of length n, where α is $(k-1)$-ary, according to the number of occurrences of the strings $1k$ and $r(r-1)$, $r>1$, in $(k-1)\alpha k$ is given by

$$WL_k(x,p,q)=(p-1)xWD_{k-1}^{(1)}(x,q|k-1)+WD_{k-1}(x,q|k-1),$$

or

$$WL_k(x,p,q)=\frac{x^{k-1}(q-1)^{k-2}(p-1)[1-x(k+q-2)]+[x+x^2(p-q)]\sum_{i=0}^{k-2}x^i(q-1)^i}{1-x(k+q-2)+x^2(q-1)\sum_{i=0}^{k-2}x^i(q-1)^i}. \tag{3.6}$$

Since each $\pi\in P(n,k)$ can be uniquely expressed as $\pi=1w^{(1)}2w^{(2)}\cdots kw^{(k)}$, where each $w^{(i)}$ is i-ary, combining (3.1), (3.2), (3.4), and (3.6) yields the following result.

Theorem 3.4. *If $k\geqslant 3$, then the generating function $J_k(x,p,q)$ for the number of partitions of $[n]$ with k blocks according to the number of occurrences of the strings $1k$ and $r(r-1)$, $r>1$, is given by*

$$J_k(x,p,q)=WE_k(x,p,q)WL_k(x,p,q)\prod_{i=1}^{k-2}WD_i(x,q|i), \tag{3.7}$$

where $WD_i(x,q|i)$, $WE_k(x,p,q)$, and $WL_k(x,p,q)$ are given, respectively, by (3.1), (3.4), and (3.6).

Taking $p = 1$ in (3.7) yields (3.3). Taking $q = 1$ in (3.7) yields the following corollary.

Corollary 3.5. *If $k \geqslant 3$, then the generating function $AD_k(x, q)$ for the number of partitions of $[n]$ with k blocks according to the number of occurrences of the string $1k$ is given by*

$$AD_k(x, q) = \frac{x^k}{(1-x)(1-2x)\cdots(1-(k-1)x)} \left(\frac{1+(q-1)x}{1-kx+(1-q)x^2} \right). \tag{3.8}$$

Taking $p = q$ in (3.7) yields the generating function which counts the members of $P(n, k)$ according to the number of circular connectors (as defined as the beginning of the section).

Corollary 3.6. *If $k \geqslant 3$, then the generating function $CD_k(x, q)$ for the number of partitions of $[n]$ with k blocks according to the number of occurrences of strings of form either $1k$ or $r(r-1)$, $r > 1$, is given by*

$$CD_k(x, q) = \frac{D_{k-2}(x, q)}{1-x(k+q-1)} \left(\frac{x^k(q-1)^{k-1}[1-x(k+q-2)] + x^2 \sum_{i=0}^{k-2} x^i(q-1)^i}{1-x(k+q-2)+x^2(q-1)\sum_{i=0}^{k-2} x^i(q-1)^i} \right), \tag{3.9}$$

where $D_{k-2}(x, q)$ is defined by (3.3).

3.2 According to the number of strings $r(r-1)\cdots(r-m)$

The idea used to establish Theorem 3.1 above may be extended to study the generating function $WD_{k,m}(x, q)$ for the number of k-ary words of length n according to the number of strings $r(r-1)\cdots(r-m)$ of arbitrary length. More precisely, let $WD_{k,m}(x, q|a_1a_2\cdots a_s)$ denote the generating function for the number of k-ary words $\pi_1\pi_2\cdots\pi_n$ of length n according to the number of strings $r(r-1)\cdots(r-m)$, with $r > m$ and $\pi_1\cdots\pi_s = a_1\cdots a_s$. Clearly, from the definitions, we have

$$WD_{k,m}(x, q) = 1 + \sum_{i=1}^{k} WD_{k,m}(x, q|i), \tag{3.10}$$

and for all i,

$$\begin{aligned}
&WD_{k,m}(x, q) \\
&\qquad = 1 + \sum_{i=1}^{k} WD_{k,m}(x, q|i), \\
&WD_{k,m}(x, q|i) \\
&\qquad = x(WD_{k,m}(x, q) - WD_{k,m}(x, q|i-1)) + WD_{k,m}(x, q|i(i-1)), \\
&WD_{k,m}(x, q|i(i-1)) \\
&\qquad = x^2(WD_{k,m}(x, q) - WD_{k,m}(x, q|i-2)) + WD_{k,m}(x, q|i(i-1)(i-2)), \\
&\qquad \vdots \\
&WD_{k,m}(x, q|i(i-1)\cdots(i-m+1)) \\
&\qquad = x^m(WD_{k,m}(x, q) - WD_{k,m}(x, q|i-m)) + WD_{k,m}(x, q|i(i-1)\cdots(i-m)), \\
&WD_{k,m}(x, q|i(i-1)\cdots(i-m)) \\
&\qquad = xqW_{k,m}(x, q|(i-1)(i-2)\cdots(i-m)).
\end{aligned} \tag{3.11}$$

In order to find an explicit formula for $WD_{k,m}(x,q)$, we will need the following lemmas.

Lemma 3.7. *Fix $m \geqslant 2$ and let $\mathbf{A}_{d,m} = (a_{ij})$ be the $d \times d$ matrix, where*

$$a_{ij} = \begin{cases} x^{i-j+1}, & \text{if } i-m+1 \leqslant j \leqslant i+1; \\ x^{i-j+1}q^{i-m-j+1}, & \text{if } 1 \leqslant j \leqslant i-m; \\ 0, & \text{otherwise.} \end{cases}$$

Then the generating function $\mathbf{A}_m(y) = \sum_{d \geqslant 0} \det(\mathbf{A}_{d,m})y^d$ is given by

$$\frac{(1+xy)(1+qxy)}{1+qxy+(q-1)(-xy)^{m+1}}.$$

Moreover, the determinant of the matrix $\mathbf{A}_{d,m}$ is given by

$$\begin{aligned} \det(\mathbf{A}_{d,m}) &= \sum_{j \geqslant 0} \binom{d-jm}{j} q^{d-jm-j}(1-q)^j(-x)^d \\ &+ (1+q)\sum_{j \geqslant 0} \binom{d-1-jm}{j} q^{d-1-jm-j}(1-q)^j(-x)^{d-1} \\ &+ q\sum_{j \geqslant 0} \binom{d-2-jm}{j} q^{d-2-jm-j}(1-q)^j(-x)^{d-2}. \end{aligned} \tag{3.12}$$

Proof. It is not hard to show that the determinant of the matrix $\mathbf{A}_{d,m}$ satisfies

$$\det(\mathbf{A}_{d,m}) = -\sum_{i=1}^{m}(-x)^i \det(\mathbf{A}_{d-i,m}) - \sum_{i=m+1}^{d}(-x)^i q^{i-m} \det(\mathbf{A}_{d-i,m}),$$

with $\det(\mathbf{A}_{0,m}) = 1$, $\det(\mathbf{A}_{1,m}) = x$ and $\det(\mathbf{A}_{j,m}) = 0$ for all $j = 2, \ldots, m$. Multiplying this recurrence by y^d and summing over all possible $d \geqslant m+1$, we obtain

$$\begin{aligned} \mathbf{A}_m(y) - 1 - xy = xy(\mathbf{A}_m(y) - 1 - xy)\left(\frac{1-(-xy)^{m-1}}{1+xy}\right) - (-xy)^m(\mathbf{A}_m(y) - 1) \\ - q\mathbf{A}_m(y)\frac{(-xy)^{m+1}}{1+qxy}, \end{aligned}$$

which is equivalent to

$$\mathbf{A}_m(y) = \frac{(1+xy)(1+qxy)}{1+qxy+(q-1)(-xy)^{m+1}}.$$

Finding the coefficient of y^d in

$$\begin{aligned} \mathbf{A}_m(y) &= (1+xy)(1+qxy)\sum_{i \geqslant 0}(x(q-1)(-xy)^m - qx)^i y^i \\ &= (1+xy)(1+qxy)\sum_{i \geqslant 0}\sum_{j=0}^{i}\binom{i}{j} q^{i-j}(1-q)^j(-x)^{i+jm}y^{i+jm} \end{aligned}$$

yields

$$\det(\mathbf{A}_{d,m}) = \sum_{j\geqslant 0}\binom{d-jm}{j}q^{d-jm-j}(1-q)^j(-x)^d$$
$$+(1+q)\sum_{j\geqslant 0}\binom{d-1-jm}{j}q^{d-1-jm-j}(1-q)^j(-x)^{d-1}$$
$$+q\sum_{j\geqslant 0}\binom{d-2-jm}{j}q^{d-2-jm-j}(1-q)^j(-x)^{d-2},$$

as claimed. □

Lemma 3.8. *Fix $m \geqslant 2$ and let $\mathbf{B}_{d,m} = \mathbf{B}_{d,m}(\alpha_1,\ldots,\alpha_d) = (b_{ij})$ be the $d\times d$ matrix, where*

$$b_{ij} = \begin{cases} x^{i-j}, & \text{if } i-m \leqslant j \leqslant i \text{ and } j < d;\\ x^{i-j}q^{i-j-m}, & \text{if } 1 \leqslant j \leqslant i-m-1;\\ \alpha_i, & \text{if } j = d;\\ 0, & \text{otherwise.}\end{cases}$$

Then the determinant of the matrix $\mathbf{B}_{d,m}$ is given by

$$\sum_{j=1}^{d}(-1)^{d+j}\alpha_j\det(\mathbf{A}_{d-j,m}),$$

where the determinant $\det(\mathbf{A}_{d-j,m})$ is given in Lemma 3.7.

Proof. This follows from expanding the determinant along the final column. □

Now we can find the generating function $WD_{k,m}(x,q)$ for $m \geqslant 2$. Induction on i, using (3.11), yields

$$WD_{k,m}(x,q|i(i-1)\cdots(i-(m-1))) = x^m\sum_{j=0}^{i-m}(xq)^j(WD_{k,m}(x,q) - WD_{k,m}(x;q|i-m-j)),$$

for all $i \geqslant m$ (take $WD_{k,m}(x,q|j)$ to be zero if $j<1$). Again using (3.11), we obtain

$$WD_{k,m}(x,q|i)$$
$$= \sum_{j=1}^{m-1}x^j(WD_{k,m}(x,q) - WD_{k,m}(x,q|i-j)) + x^m\sum_{j=0}^{i-m}(xq)^j(WD_{k,m}(x,q) - WD_{k,m}(x,q|i-m-j)),$$

for all $i \geqslant m$, with the initial conditions $WD_{k,m}(x,q|i) = xWD_{k,m}(x,q)$ for $i = 1,2,\ldots,m-1$. We may rewrite these equations for $m \leqslant i \leqslant k$ in matrix form as

$$\mathbf{WD}_{k;m}\begin{pmatrix} WD_{k,m}(x,q|m)\\ \vdots \\ WD_{k,m}(x,q|k)\end{pmatrix} = \begin{pmatrix}\beta_m\\ \vdots\\ \beta_k\end{pmatrix}WD_{k,m}(x,q), \tag{3.13}$$

where $\mathbf{WD}_{k,m} = (w_{ij})_{1\leqslant i,j\leqslant k-m+1}$ is a $(k-m+1)\times(k-m+1)$ matrix defined by

$$w_{ij} = \begin{cases} x^{i-j}, & \text{if } i-m \leqslant j \leqslant i; \\ x^{i-j}q^{i-j-m}, & \text{if } 1 \leqslant j \leqslant i-m-1; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\beta_i = \sum_{j=1}^{m-1} x^j + x^m \sum_{j=0}^{i-m}(xq)^j - x\sum_{j=i-m+1}^{m-1} x^j - x^{m+1}\sum_{j=\max\{i-2m+1,\ 0\}}^{i-m-1}(xq)^j,\ \ i=m,m+1,\ldots,k. \tag{3.14}$$

Theorem 3.9. *If $k \geqslant m \geqslant 2$, then the generating function $WD_{k,m}(x,q)$ is given by*

$$\frac{1}{1-(m-1)x-\sum_{i=0}^{k-m}\sum_{j=0}^{i}(-1)^{i+j}\beta_{m+j}\det(\mathbf{A}_{i-j,m})}.$$

Furthermore, the generating function $WD_{k,m}(x,q|i)$ is given by $xWD_{k,m}(x,q)$ for $i=1,2,\ldots,m-1$ and

$$WD_{k,m}(x,q|i) = WD_{k,m}(x,q)\sum_{j=1}^{i+1-m}(-1)^{i-1-m+j}\beta_{m-1+j}\det(\mathbf{A}_{i+1-m-j,m})$$

for $i=m,m+1,\ldots,k$, where β_i is defined by (3.14).

Proof. Cramer's rule, together with Lemmas 3.7 and 3.8, yields

$$\begin{aligned} WD_{k,m}(x,q|i) &= \det(\mathbf{B}_{i+1-m,m}(\beta_m,\ldots,\beta_i))WD_{k,m}(x,q) \\ &= WD_{k,m}(x,q)\sum_{j=1}^{i+1-m}(-1)^{i-1-m+j}\beta_{m-1+j}\det(\mathbf{A}_{i+1-m-j,m}), \end{aligned} \tag{3.15}$$

where $i=m,m+1,\ldots,k$, and

$$\begin{aligned} \det(\mathbf{A}_{d,m}) &= \sum_{j\geqslant 0}\binom{d-jm}{j}q^{d-jm-j}(1-q)^j(-x)^d \\ &+ (1+q)\sum_{j\geqslant 0}\binom{d-1-jm}{j}q^{d-1-jm-j}(1-q)^j(-x)^{d-1} \\ &+ q\sum_{j\geqslant 0}\binom{d-2-jm}{j}q^{d-2-jm-j}(1-q)^j(-x)^{d-2}. \end{aligned}$$

Combining (3.10) and (3.15), and noting $WD_{k,m}(x,q|i) = xWD_{k,m}(x,q)$ for all $i=1,2,\ldots,m-1$, yields the desired result. ∎

Since each set partition π of $[n]$ with exactly k blocks can be decomposed as $\pi = 1w^{(1)}2w^{(2)}\cdots kw^{(k)}$, where each $w^{(i)}$ is i-ary, we see that the generating function $D_{k,m}(x,q)$ for the number of set partitions of $[n]$ according to the number of strings $r(r-1)\cdots(r-m)$ with $r>m$ is given by

$$D_{k,m}(x,q) = \prod_{i=1}^{k} WD_{i,m}(x,q|i),$$

which, by the above theorem, yields the following result.

Theorem 3.10. *If $m \geqslant 2$, then the generating function for the number of partitions of $[n]$ with k blocks according to the number of occurrences of the string $r(r-1)\cdots(r-m)$ with $r > m$ is given by*

$$D_{k,m}(x,q) = \prod_{i=1}^{k} WD_{i,m}(x,q|i),$$

where $WD_{i,m}(x,q|i) = \frac{x}{1-ix}$ when $i < m$ and is given by Theorem 3.9 when $i \geqslant m$.

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